Chapter 2

Fundamental solution and Green’s functions for two-dimensional problem in orthotropic thermoelastic diffusion media

2.1 Introduction

Fundamental solutions or Green’s functions play an important role in the solution of numerous problems in the mechanics and physics of solids. They are a basic building block of many further works. For example, fundamental solutions can be used to construct many analytical solutions of practical problems when boundary conditions are imposed. They are essential in the boundary element method as well as the study of cracks, defects and inclusions.

Ding, Chen and Liang (1996) derived the general solutions for coupled equations in piezoelectric media. Dunn and Wienecke (1999) derived the half space Green’s functions for transversely isotropic piezoelectric solid. Pan and Tanon (2000) studied the Green's
functions for three dimensional problem in general anisotropic piezoelectric solids. Chen (2000) presented a general solution for transverse isotropic thermo-piezo-elastic media in terms of five harmonic functions and derived an exact solution for a penny shaped cracked subjected to uniform temperature load.


In the present chapter, the fundamental solution and Green’ function for two-dimensional problem in orthotropic thermoelastic diffusion media are investigated. After applying the dimensionless quantities and using the operator theory, the general expression for components of displacement, stress, temperature change and mass concentration are derived. On the basis of general solution, the fundamental solution and Green’s functions for a steady line heat source in a semi-infinite orthotropic thermoelastic diffusion media are derived. The components of displacement, temperature change and mass concentration are computed numerically and presented graphically. All the components are expressed in terms of elementary functions, so it is convenient to use them. Some special cases are also deduced.
2.2 Basic equations

Following Sherief and Saleh (2005), the basic equations in homogeneous anisotropic thermoelastic diffusion media in the absence of body forces, heat sources and mass diffusion sources are

Constitutive relations

\[ \sigma_{ij} = c_{ijkl} e_{km} + a_{ij} T + b_{ij} C, \]  
\[ (2.2.1) \]

Equations of motion

\[ \sigma_{ij,j} = \rho \ddot{u}_i, \]  
\[ (2.2.2) \]

Equation of heat conduction

\[ \rho C_E \dot{T} + a T_0 \dot{C} - a_{ij} T_0 \dot{e}_{ij} = K_{ij} T_{ij}, \]  
\[ (2.2.3) \]

Equation of mass diffusion

\[ -\alpha_{ij}^* b_{km} e_{m,ij} - \alpha_{ij}^* b C_{,ij} + \alpha_{ij}^* a T_{ij} = -\dot{C}, \]  
\[ (2.2.4) \]

In the equations (2.2.1)-(2.2.4), \( c_{ijkl} \) \((c_{klij} = c_{jikm} = c_{ijmk})\) is the tensor of elastic constant, \( \rho \) is the density, \( C_E \) is the specific heat at constant strain, \( u_i \) are the components of displacement vector \( u \), \( \sigma_{ij} \) \((= \sigma_{ji})\) and \( e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \) are, respectively, the components of stress and strain tensor, \( K_{ij} \) \((= K_{ji})\), \( \alpha_{ij}^* \) \((= \alpha_{ji}^*)\) are, respectively, the coefficients of thermal conductivity and diffusion tensor, \( T = \Theta - T_0 \) is the small temperature increment, \( \Theta \) is the absolute temperature of the medium, \( T_0 \) is the reference temperature of the body so chosen such that \( \left| \frac{T}{T_0} \right| << 1 \), \( C \) is the concentration of the diffusion material in the body, \( a, b \) are, respectively, coefficient describing the measure of thermodiffusion and mass diffusion effects, \( a_{ij}, b_{ij} \) are, respectively the tensor of thermal and diffusion moduli. The symbol (";") followed by a suffix denotes
differentiation with respect to spatial coordinate and a superposed dot ("."\) denotes the derivative with respect to time.

The equations (2.2.1)-(2.2.4), for orthotropic thermoelastic diffusion media in Cartesian coordinate system \((x, y, z)\) in component form can be written as

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{yz} \\
\sigma_{xz} \\
\sigma_{xy}
\end{bmatrix} = \begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\
c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}
\end{bmatrix} \begin{bmatrix}
e_{xx} \\
e_{yy} \\
e_{zz} \\
e_{yz} \\
e_{xz} \\
e_{xy}
\end{bmatrix} - \begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
2e_{yz} \\
2e_{xz} \\
2e_{xy}
\end{bmatrix} T - \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
0 \\
0 \\
0
\end{bmatrix} C,
\]

(2.2.5)

\[
c_{11} \frac{\partial^2 u}{\partial x^2} + e_{66} \frac{\partial^2 u}{\partial y^2} + c_{55} \frac{\partial^2 u}{\partial z^2} + (c_{12} + e_{66}) \frac{\partial^2 v}{\partial x \partial y} + (c_{13} + c_{55}) \frac{\partial^2 w}{\partial x \partial z} - \\
a_1 \frac{\partial T}{\partial x} - b_1 \frac{\partial C}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2},
\]

(2.2.6)

\[
(c_{12} + e_{66}) \frac{\partial^2 u}{\partial x \partial y} + e_{66} \frac{\partial^2 v}{\partial x^2} + c_{22} \frac{\partial^2 v}{\partial y^2} + c_{44} \frac{\partial^2 v}{\partial z^2} + (c_{23} + c_{44}) \frac{\partial^2 w}{\partial y \partial z} - \\
a_2 \frac{\partial T}{\partial y} - b_2 \frac{\partial C}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2},
\]

(2.2.7)

\[
(c_{13} + c_{55}) \frac{\partial^2 u}{\partial x \partial z} + (c_{23} + c_{44}) \frac{\partial^2 v}{\partial y \partial z} + c_{55} \frac{\partial^2 w}{\partial x^2} + c_{44} \frac{\partial^2 w}{\partial y^2} + c_{33} \frac{\partial^2 w}{\partial z^2} - \\
a_3 \frac{\partial T}{\partial z} - b_3 \frac{\partial C}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2},
\]

(2.2.8)

\[
\rho C_E \frac{\partial T}{\partial t} + aT_0 \frac{\partial C}{\partial t} + T_0 \left[ a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} + a_3 \frac{\partial u}{\partial z} \right] = \\
K_1 \frac{\partial^2 T}{\partial x^2} + K_2 \frac{\partial^2 T}{\partial y^2} + K_3 \frac{\partial^2 T}{\partial z^2},
\]

(2.2.9)
\[ b_1 \left[ \alpha_1^* \frac{\partial^3 u}{\partial x^3} + \alpha_2^* \frac{\partial^3 u}{\partial x \partial y^2} + \alpha_3^* \frac{\partial^3 u}{\partial x \partial z^2} \right] + b_2 \left[ \alpha_1^* \frac{\partial^3 v}{\partial x^2 \partial y} + \alpha_2^* \frac{\partial^3 v}{\partial y^3} + \alpha_3^* \frac{\partial^2 v}{\partial y \partial z^2} \right] + \\
\left[ \alpha_1^* \frac{\partial^3 w}{\partial x^2 \partial z} + \alpha_2^* \frac{\partial^3 w}{\partial y^2 \partial z} + \alpha_3^* \frac{\partial^3 w}{\partial z^3} \right] + \alpha_1^* \frac{\partial^2 T}{\partial x^2} + \alpha_2^* \frac{\partial^2 T}{\partial y^2} + \alpha_3^* \frac{\partial^2 T}{\partial z^2} \right] - \\
b \left[ \alpha_1^* \frac{\partial^2 C}{\partial x^2} + \alpha_2^* \frac{\partial^2 C}{\partial y^2} + \alpha_3^* \frac{\partial^2 C}{\partial z^2} \right] = - \frac{\partial C}{\partial t} \]

(2.2.10)

where

\[ a_{ij} = -a_i \delta_{ij}, \ b_{ij} = -b_i \delta_{ij}, \ K_{ij} = K_i \delta_{ij}, \ \alpha_{ij}^* = \alpha_i^* \delta_{ij}, \quad (i \text{ is not summed}) \]

\[ a_1 = c_{11} \alpha_1 + c_{12} \alpha_2 + c_{13} \alpha_3, \ a_2 = c_{12} \alpha_1 + c_{22} \alpha_2 + c_{23} \alpha_3, \ a_3 = c_{13} \alpha_1 + c_{23} \alpha_2 + c_{33} \alpha_3, \]

\[ b_1 = c_{11} \alpha_{1c} + c_{12} \alpha_{2c} + c_{13} \alpha_{3c}, \ b_2 = (c_{12} \alpha_{1c} + c_{22} \alpha_{2c} + c_{23} \alpha_{3c}), \]

\[ b_3 = c_{13} \alpha_{1c} + c_{23} \alpha_{2c} + c_{33} \alpha_{3c}. \]

Here \( \alpha_t \) and \( \alpha_{tc} \) are respectively the coefficient of linear thermal expansion and diffusion expansion. In the equations (2.2.5)-(2.2.10), we use the contracting subscript notation \( 11 \rightarrow 1, \ 22 \rightarrow 2, \ 33 \rightarrow 3, \ 23 \rightarrow 4, \ 13 \rightarrow 5, \ 12 \rightarrow 6 \) to relate \( c_{ijkm} \) to \( c_{\delta\ell} (i, j, k, m = 1, 2, 3 \) and \( \delta, \ell = 1, 2, 3, 4, 5, 6) \).

### 2.3 Formulation and solution of the problem

We consider a homogenous orthotropic thermoelastic diffusion medium. Let us take \( Oxyz \) as the frame of reference in Cartesian coordinates, the origin \( O \) being any point on the plane boundary.

For two-dimensional static problem, we assume the displacement vector, temperature change and mass concentration respectively, of the form
\[ u = (u(x, z), 0, w(x, z)), \quad T(x, z), \quad C(x, z). \]  

(2.3.1)

Making use of equation (2.3.1) in equations (2.2.6)-(2.2.10), we obtain

\[ c_{11} \frac{\partial^2 u}{\partial x^2} + c_{55} \frac{\partial^2 u}{\partial z^2} + (c_{13} + c_{55}) \frac{\partial^2 w}{\partial x \partial z} - a_1 \frac{\partial T}{\partial x} - b_1 \frac{\partial C}{\partial x} = 0, \]

(2.3.2)

\[ (c_{13} + c_{55}) \frac{\partial^2 u}{\partial x \partial z} + c_{55} \frac{\partial^2 w}{\partial x^2} + c_{33} \frac{\partial^2 w}{\partial z^2} - a_3 \frac{\partial T}{\partial z} - b_3 \frac{\partial C}{\partial z} = 0, \]

(2.3.3)

\[ K_1 \frac{\partial^2 T}{\partial x^2} + K_3 \frac{\partial^2 T}{\partial z^2} = 0, \]

(2.3.4)

\[ b_1 \left[ \alpha_1^* \frac{\partial^3 u}{\partial x^3} + \alpha_3^* \frac{\partial^3 u}{\partial x^2 \partial z} \right] + b_3 \left[ \alpha_1^* \frac{\partial^3 w}{\partial x^2 \partial z} + \alpha_3^* \frac{\partial^3 w}{\partial z^3} \right] 
+ a \left[ \alpha_1^* \frac{\partial^2 T}{\partial x^2} + \alpha_3^* \frac{\partial^2 T}{\partial z^2} \right] - b \left[ \alpha_1^* \frac{\partial^2 C}{\partial x^2} + \alpha_3^* \frac{\partial^2 C}{\partial z^2} \right] = 0. \]

(2.3.5)

We define the dimensionless quantities as

\[ (x', z', u', w') = \frac{\omega_1^*}{v_1} (x, z, u, w), \quad (T', C') = \frac{1}{c_{11}} (a_1 T, b_1 C). \]

\[ \sigma_{ij}' = \frac{\sigma_{ij}}{a_1 T_0}, \quad H' = \frac{a_1 v_1}{c_{11} K_1 \omega_1^*} H, \]

(2.3.6)

where

\[ v_1^2 = b_1, \quad \omega_1^* = \frac{ac_{11}}{K_1}, \]

and \( K_1 \) and \( b_1 \) are, respectively, coefficient of thermal conductivity and diffusion moduli.

Using the dimensionless quantities defined by equation (2.3.6) in equations (2.3.2)-(2.3.5), after suppressing the primes, we obtain
\[
\frac{\partial^2 u}{\partial x^2} + \delta_1 \frac{\partial^2 u}{\partial z^2} + \delta_2 \frac{\partial^2 w}{\partial \xi \partial z} - \left( \frac{\partial}{\partial x} \right) T - \left( \frac{\partial}{\partial x} \right) C = 0,
\]
(2.3.7)

\[
\delta_2 \frac{\partial^2 u}{\partial x \partial z} + \delta_1 \frac{\partial^2 w}{\partial x^2} + \delta_3 \frac{\partial^2 w}{\partial z^2} - \varepsilon_1 \left( \frac{\partial}{\partial z} \right) T - \varepsilon_2 \left( \frac{\partial}{\partial z} \right) C = 0,
\]
(2.3.8)

\[
\left( \frac{\partial^2 T}{\partial x^2} + \varepsilon_3 \frac{\partial^2 T}{\partial z^2} \right) = 0,
\]
(2.3.9)

\[
\left( q_1 \frac{\partial^3 u}{\partial x^3} + q_2 \frac{\partial^3 u}{\partial x^2 \partial z} \right) + \left( q_3 \frac{\partial^2 w}{\partial x^2 \partial z} + q_4 \frac{\partial^2 w}{\partial z^3} \right) + \\
+ \left( q_5 \frac{\partial^2 T}{\partial x^2} + q_6 \frac{\partial^2 T}{\partial z^2} \right) - \left( q_7 \frac{\partial^2 C}{\partial x^2} + q_8 \frac{\partial^2 C}{\partial z^2} \right) = 0,
\]
(2.3.10)

Where

\[
(\delta_1, \delta_2, \delta_3) = \frac{1}{c_{11}} (c_{55}, c_{13} + c_{55}, c_{33}), \varepsilon_1 = \frac{a_1}{b_1}, \varepsilon_2 = \frac{b_1}{b_1}, \varepsilon_3 = \frac{K_3}{K_1},
\]

\[
(q_1, q_2, q_3, q_4) = \frac{1}{c_{11}} \left( \alpha_1 \omega_1^* b_1, \alpha_3 \omega_1^* b_1, \alpha_1 \omega_1^* b_3, \alpha_3 \omega_1^* b_3 \right),
\]

\[
(q_5, q_6) = \frac{1}{a_1} \left( \alpha_1 \omega_1^* a, \alpha_3 \omega_1^* a \right), \quad (q_7, q_8) = \frac{1}{b_1} \left( \alpha_1 \omega_1^* b, \alpha_3 \omega_1^* b \right).
\]

The equations (2.3.7)-(2.3.10) can be written as

\[
D(u, w, C, T)_{ij}^{tr} = 0,
\]
(2.3.11)

where \( D \) is differential operator matrix given by
Equation (2.3.11) is a homogeneous set of differential equations in \( u, w, C, T \). The general solution by the operator theory as follows

\[
 u = A_{i1} F, \quad w = A_{i2} F, \quad C = A_{i3} F, \quad T = A_{i4} F, \quad (i = 1, 2, 3, 4) \quad (2.3.13)
\]

where \( A_{ij} (i, j = 1, 2, 3, 4) \) are algebraic cofactors of the matrix D, of which the determinant is

\[
|D| = \left( a^* \frac{\partial^6}{\partial z^6} + b^* \frac{\partial^6}{\partial x^2 \partial z^4} + c^* \frac{\partial^6}{\partial x^4 \partial z^2} + d^* \frac{\partial^6}{\partial x^6} \right) \times \left( \frac{\partial^2}{\partial x^2} + \epsilon_3 \frac{\partial^2}{\partial z^2} \right). \quad (2.3.14)
\]

and

\[
a^* = \delta_1 (\epsilon_2 q_4^* - \delta_3 q_8^*), \quad b^* = \delta_1 (\epsilon_2 q_4^* - \delta_3 q_8^*), \quad c^* = \delta_1 (q_2^* - q_8^*), \quad d^* = \delta_1 (q_1^* - q_7^*).
\]

The function \( F \) in equation (2.3.13) satisfies the following homogeneous equation

\[
|D|F = 0. \quad (2.3.15)
\]

It can be seen that if \( i = 1, 2, 3 \) are taken in equation (2.3.13), three general solutions are obtained in which \( T = 0 \). These solutions are identical to those without thermal effect and are not discussed here. Therefore if \( i = 4 \) should be taken in equation (2.3.13), the following solution is obtained

\[
\begin{bmatrix}
\frac{\partial^2}{\partial x^2} + \delta_1 \frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial x \partial z} & -\frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\
\delta_2 \frac{\partial^2}{\partial x \partial z} & \delta_1 \frac{\partial^2}{\partial x^2} + \delta_3 \frac{\partial^2}{\partial z^2} & -\epsilon_2 \frac{\partial}{\partial z} & -\epsilon_1 \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} \left( q_1^* \frac{\partial^2}{\partial x^2} + q_2^* \frac{\partial^2}{\partial z^2} \right) & \frac{\partial}{\partial z} \left( q_3^* \frac{\partial^2}{\partial x^2} + q_4^* \frac{\partial^2}{\partial z^2} \right) & -\left( q_7^* \frac{\partial^2}{\partial x^2} + q_8^* \frac{\partial^2}{\partial z^2} \right) & \left( q_3^* \frac{\partial^2}{\partial x^2} + q_8^* \frac{\partial^2}{\partial z^2} \right) \\
0 & 0 & 0 & \left( \frac{\partial^2}{\partial x^2} + \epsilon_3 \frac{\partial^2}{\partial z^2} \right)
\end{bmatrix}.
\]

(2.3.12)
where

\[ p_1 = -(q_3^* + q_7^*) \delta_1, \quad q_1 = (\varepsilon_2 - \varepsilon_1)q_3^* + (\varepsilon_2 q_5^* + \varepsilon_1 q_7^*) \delta_2 - \delta_1 (q_8^* + q_6^*) - \delta_3 (q_5^* + q_7^*). \]

\[ r_1 = (\varepsilon_1 q_8^* + \varepsilon_2 q_6^*) \delta_2 - (q_6^* + q_8^*) \delta_3 - (\varepsilon_1 - \varepsilon_2) q_4^*, \quad p_2 = (q_1^* - q_7^*) \varepsilon_1 + (q_7^* + q_5^*) \delta_2 - (q_1^* + q_5^*) \varepsilon_2. \]

\[ r_2 = -(\varepsilon_1 q_8^* + \varepsilon_2 q_6^*) \delta_1, \quad q_2 = (\delta_2 - \varepsilon_2)q_6^* + (q_2^* - q_8^*) \varepsilon_1 - \delta_1 (q_1^* q_7^* + q_2^* q_5^*) + \delta_2 q_8^* - \varepsilon_2 q_2^*. \]

\[ p_3 = -(\delta_3 q_6^* + \varepsilon_1 q_4^*) \delta_1, \quad q_3 = (\varepsilon_1 - \delta_2)q_4^* - (q_2^* + q_6^*) \delta_3 - (\delta_1^2 - \delta_2^2) q_5^* - (\delta_3 q_5^* + \varepsilon_1 q_3^*) \delta_1 + \varepsilon_1 \delta_2 q_2^*. \]

\[ r_3 = (\delta_2 - \varepsilon_1)q_3^* - (q_6^* + q_2^*) \delta_1 - (\delta_1^2 - \delta_2^2) q_5^* - (q_1^* + q_5^*) \delta_3 - \varepsilon_1 \delta_2 q_1^*. \]

\[ l_3 = -(q_5^* + q_1^*) \delta_1. \]

Equation (2.3.15) can be rewritten as

\[
\prod_{j=1}^{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2} \right) F = 0, \quad (2.3.17)
\]

where \( z_j = s_j z, \quad s_4 = \sqrt{\frac{K_1}{K_3}}, \) and \( s_j (j = 1, 2, 3) \) are three roots (with positive real part) of the following algebraic equation

\[
a^* s^6 - b^* s^4 + c^* s^2 - d^* = 0. \quad (2.3.18)
\]
The generalized Almansi theorem [Ding et al., (1996)], stated below

**Statement:** Let $R$ be region of the $(x, y, z)$ space such that a straight line parallel to the $z$-axis intersects the boundary of $R$ at no more than two points. Let $F_n(x, y, z)$ be a solution of

$$\prod_{i=1}^{n} \nabla_i^2 F_n = \nabla_1^2 \nabla_2^2 \nabla_3^2 \ldots \nabla_{n-1}^2 \nabla_n^2 F_n = 0 \text{ in } R$$

where

$$\nabla_i^2 = \Lambda + c_i^2 D^2, \quad \Lambda = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad D = \frac{\partial}{\partial z}$$

and $c_i$ are constant. Then $F_n$ can be represented as

$$F_n(x, y, z) = F_{n-1}(x, y, z) + z^m F^{(n)}(x, y, z),$$

where $F_{n-1}$ and $F^{(n)}$ are, respectively, satisfy

$$\prod_{i=1}^{n-1} \nabla_i^2 F_{n-1} = 0,$$

$$\nabla_n^2 F^{(n)} = 0,$$

and $m(0 \leq m \leq n - 1)$ is the number of coefficients $c_i^2 (i = 1, 2, 3, \ldots, n - 1)$ which are equal to $c_n^2$.

Use the above theorem, the function $F$ can be expressed in terms of four harmonic functions

(I) $F = F_1 + F_2 + F_3 + F_4$ for distinct $s_j (j = 1, 2, 3, 4)$, \hspace{1cm} (2.3.19)

(II) $F = F_1 + F_2 + F_3 + zF_4$ for $s_1 \neq s_2 \neq s_3 = s_4$, \hspace{1cm} (2.3.20)

(III) $F = F_1 + F_2 + zF_3 + z^2 F_4$ for $s_1 \neq s_2 = s_3 = s_4$, \hspace{1cm} (2.3.21)
\[(IV)\quad F = F_1 + zF_2 + z^2F_3 + z^3F_4, \quad \text{for} \quad s_1 = s_2 = s_3 = s_4, \quad (2.3.22)\]

where \(F_j\) satisfies the following harmonic equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2} \right) F_j = 0, \quad (j = 1,2,3,4). \tag{2.3.23}\]

The general solution for the case of distinct roots, can be derived as follows

\[
u = \sum_{j=1}^{4} p_{1j} \frac{\partial^4 F_j}{\partial z_j^4}, \quad w = \sum_{j=1}^{4} s_j p_{2j} \frac{\partial^5 F_j}{\partial z_j^5}, \quad C = \sum_{j=1}^{4} p_{3j} \frac{\partial^6 F_j}{\partial z_j^6}, \quad T = p_{44} \frac{\partial^6 F_4}{\partial z_4^6}, \tag{2.3.24}\]

where

\[
p_{kj} = p_k - q_k s_j^2 + r_k s_j^4, \]
\[
p_{3j} = p_3 s_j^6 - q_3 s_j^4 + r_3 s_j^2 - 1, \quad (k = 1,2) \text{ and } j = (1,2,3) \]
\[
p_{44} = a^* s_4^6 - b^* s_4^4 + c^* s_4^2 - d^*. \]

The general solution for the other three cases can be derived in the similar way.

Equation (2.3.24) can be further simplified by taking

\[
p_{1j} \frac{\partial^4 F_j}{\partial z_j^4} = \psi_j. \tag{2.3.25}\]

Making use of (2.3.25) in equation (2.3.24) gives

\[
u = \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial x}, \quad w = \sum_{j=1}^{4} s_j P_{1j} \frac{\partial \psi_j}{\partial z_j}, \quad C = \sum_{j=1}^{4} P_{2j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad T = P_{34} \frac{\partial^2 \psi_4}{\partial z_4^2}, \tag{2.3.26}\]

where

\[
P_{1j} = p_{2j} / p_{1j}, \quad P_{2j} = p_{3j} / p_{1j}, \quad P_{34} = p_{44} / p_{14}. \]
The function $\psi_j$ satisfies the harmonic equations

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2} \right) \psi_j = 0, \quad j = 1,2,3,4.
$$

(2.3.27)

Making use of (2.3.1) in equation (2.2.5) and applying the dimensionless quantities defined by (2.3.6) on resulting equations, after suppressing the primes and with the aid of (2.3.26), we obtain

$$
\sigma_{xx} = \sum_{j=1}^{4} \left( -f_1 + f_2 s_j^2 P_{1j} - f_4 P_{3j} - f_1 P_{2j} \right) \frac{\partial^2 \psi_j}{\partial z_j^2},
$$

(2.3.28)

$$
\sigma_{zz} = \sum_{j=1}^{4} \left( -f_2 + h_1 s_j^2 P_{1j} - h_2 P_{3j} - h_3 P_{2j} \right) \frac{\partial^2 \psi_j}{\partial z_j^2},
$$

(2.3.29)

$$
\sigma_{zz} = \sum_{j=1}^{4} h_4 (1 + P_{1j}) s_j \frac{\partial^2 \psi_j}{\partial x \partial z_j},
$$

(2.3.30)

where

$$
P_{31} = P_{32} = P_{33} = 0,
$$

and

$$
f_1 = \frac{c_{11}}{a_1 T_0}, f_2 = \frac{c_{13}}{a_1 T_0}, h_1 = \frac{c_{33}}{a_1 T_0}, h_2 = \frac{a_1 f_1}{a_1}, h_3 = \frac{b_1 f_1}{b_1}, h_4 = \frac{c_{35}}{a_1 T_0}.
$$

Making use of equation (2.3.1) in equation (2.2.2) and with the aid of equations (2.3.6), (2.3.28)-(2.3.30), we obtain

$$
f_1 - f_2 s_j^2 P_{1j} + f_4 P_{3j} + f_1 P_{2j} = h_4 (1 + P_{1j}) s_j^2,
$$

(2.3.31)

$$
-f_2 + h_1 s_j^2 P_{1j} - h_2 P_{3j} - h_3 P_{2j} = h_4 (1 + P_{1j}),
$$

(2.3.32)

$$
(1 - \varepsilon_j s_j^2) P_{3j} = 0, \quad (j = 1,2,3,4).
$$

(2.3.33)
The values of $\sigma_{zz}, \sigma_{xx}$ and $\sigma_{zx}$ given by equations (2.3.28)-(2.3.30) with the help of equations (2.3.31)-(2.3.32) can be written as

$$
\sigma_{xx} = -\sum_{j=1}^{4} w_{ij} s_j \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad \sigma_{zz} = \sum_{j=1}^{4} w_{ij} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad \sigma_{zx} = \sum_{j=1}^{4} w_{ij} s_j \frac{\partial^2 \psi_j}{\partial x \partial z_j},
$$

(2.3.34)

where

$$
w_{ij} = f_1 - P_{1j} s_j^2 f_2 + P_{3j} f_1 + P_{2j} f_1 = h_j (1 + P_{1j}) = -f_2 + P_{1j} s_j^2 - P_{3j} h_2 - P_{3j} h_3.
$$

(2.3.35)

2.4 Fundamental solution for a line heat source on the surface of a semi-infinite orthotropic thermoelastic diffusion material

We consider a semi infinite orthotropic thermoelastic diffusion material $z \geq 0$. A line heat source $H$ is applied at the origin of two-dimensional Cartesian coordinate $(x, z)$ and the surface $z = 0$ has been taken as the stress free, thermally insulated and impermeable boundary. The complete geometry of the problem is shown in figure 2.1. The general solution given by equations (2.3.26) and (2.3.34) is derived in this section.

Following Hou et al. (2011), introducing the harmonic functions

$$
\psi_j = A_j \left[ \frac{1}{2} \left( z_j^2 - x^2 \right) \log r_j - \frac{3}{2} \right] - x z_j \tan^{-1} \frac{x}{z_j},
$$

(2.4.1)

where $A_j (j = 1, 2, 3, 4)$ are arbitrary constants to be determined and

$$
r_j = \sqrt{x^2 + z_j^2}, \quad (j = 1, 2, 3, 4).
$$

(2.4.2)
The boundary conditions at the surface \((z = 0)\) are of the form

(i) Stress free boundary

\[
\sigma_{zz} = \sigma_{zx} = 0, \quad (2.4.3)
\]

(ii) Insulated boundary

\[
\frac{\partial T}{\partial z} = 0, \quad (2.4.4)
\]

(iii) Impermeable boundary

\[
\frac{\partial C}{\partial z} = 0. \quad (2.4.5)
\]

Considering the mechanical, thermal and concentration equilibrium for a rectangle of

\(0 \leq z \leq \bar{a}\) and \(-\bar{b} \leq x \leq \bar{b}\) \((\bar{b} > 0)\), \(\text{Figure 2.1}\) three equations can be obtained

\[
\int_{-\bar{b}}^{\bar{b}} \sigma_{zz}(x, \bar{a}) dx + \int_{0}^{\bar{a}} \left[ \sigma_{zx}(\bar{b}, z) - \sigma_{zx}(-\bar{b}, z) \right] dz = 0, \quad (2.4.6)
\]

\[
- \varepsilon_3 \int_{-\bar{b}}^{\bar{b}} \frac{\partial T}{\partial z}(x, \bar{a}) dx - \int_{0}^{\bar{a}} \left[ \frac{\partial T}{\partial x}(\bar{b}, z) - \frac{\partial T}{\partial x}(-\bar{b}, z) \right] dz = H, \quad (2.4.7)
\]

\[
\int_{-\bar{b}}^{\bar{b}} \frac{\partial C}{\partial z}(x, \bar{a}) dx + \int_{0}^{\bar{a}} \left[ \frac{\partial C}{\partial x}(\bar{b}, z) - \frac{\partial C}{\partial x}(-\bar{b}, z) \right] dz = 0. \quad (2.4.8)
\]

Substituting the values of \(\psi_j (j = 1,2,3,4)\) from equation (2.4.1) in equations (2.2.26) and (2.3.32), we obtain

\[
u = - \sum_{j=1}^{4} A_j \left[ x(\log r_j - 1) + z_j \tan^{-1} \frac{x}{z_j} \right], \quad (2.4.9)
\]

\[
w = \sum_{j=1}^{4} s_j P_{1j} A_j \left[ z_j (\log r_j - 1) - x \tan^{-1} \frac{x}{z_j} \right], \quad (2.4.10)
\]
\[ T = A_4 P_{34} \log r_4, \]  
\[ (2.4.11) \]

\[ C = \sum_{j=1}^{4} A_j P_{2j} \log r_j, \]  
\[ (2.4.12) \]

\[ \sigma_{xx} = -\sum_{j=1}^{4} s_j^2 w_{1j} A_j \log r_j, \]  
\[ (2.4.13) \]

\[ \sigma_{zz} = \sum_{j=1}^{4} w_{1j} A_j \log r_j, \]  
\[ (2.4.14) \]

\[ \sigma_{zx} = -\sum_{j=1}^{4} s_j w_{1j} A_j \tan^{-1} \frac{x}{z_j}. \]  
\[ (2.4.15) \]

Substituting the values of \( \sigma_{zz}, \sigma_{xx}, T, C \) from equations (2.4.11), (2.4.12), (2.4.14), and (2.4.15) in equation (2.4.3)-(2.4.5), we obtain

\[ \sum_{j=1}^{4} w_{1j} A_j = 0, \]  
\[ (2.4.16) \]

\[ \sum_{j=1}^{4} s_j w_{1j} A_j = 0, \]  
\[ (2.4.17) \]

\[ zs_4^2 P_{34} A_4 = 0. \]  
\[ (2.4.18) \]

\[ \sum_{j=1}^{4} \frac{z s_j^2 P_{2j} A_j}{x^2 + z_j^2} = 0. \]  
\[ (2.4.19) \]

Equations (2.4.18) and (2.4.19) are satisfied automatically at the surface \( z = 0 \).

Making use the values of \( \sigma_{zz}, \sigma_{xx} \) from equations (2.4.14) and (2.4.15) in equation (2.4.6) yields
\[
\sum_{j=1}^{4} w_{1j} A_j I_3 = 0, \tag{2.4.20}
\]

where

\[
I_3 = \left[ x \left( \log \sqrt{x^2 + s_j^2 a_j^2} - 1 \right) + s_j a \tan^{-1} \frac{x}{s_j a} \right]_{x=-b}^{x=b} - 2 \left[ z_j \tan^{-1} \frac{\bar{b}}{z_j} + \bar{b} \log \sqrt{\bar{b}^2 + z_j^2} \right]_{z=0}^{z=\bar{a}} = 2\bar{b} (\log \bar{b} - 1). \tag{2.4.21}
\]

By virtue of the equation \((2.4.21)\), equation \((2.4.20)\) degenerate to equation \((2.4.16)\) i.e. equations \((2.4.6)\) and \((2.4.20)\) are satisfied automatically.

Some useful integrals are listed as follows

\[
\int \frac{\partial T}{\partial z} \, dx = s_4 P_{34} A_4 \int \frac{z_4}{r_4^2} \, dx = s_4 P_{34} A_4 \tan^{-1} \frac{x}{z_4}, \tag{2.4.22}
\]

\[
\int \frac{\partial T}{\partial x} \, dx = P_{34} A_4 \int \frac{x}{r_4^2} \, dx = -\frac{P_{34}}{s_4} A_4 \tan^{-1} \frac{x}{z_4}, \tag{2.4.23}
\]

\[
\int \frac{\partial C}{\partial z} \, dx = \sum_{j=1}^{4} A_j s_j^2 P_{2j} \int \frac{z}{x^2 + s_j^2 z^2} \, dx = \sum_{j=1}^{4} A_j s_j P_{2j} \tan^{-1} \frac{x}{s_j z}, \tag{2.4.24}
\]

\[
\int \frac{\partial C}{\partial x} \, dz = \sum_{j=1}^{4} A_j P_{2j} \int \frac{x}{x^2 + s_j^2 z^2} \, dz = -\sum_{j=1}^{4} \frac{A_j}{s_j} P_{2j} \tan^{-1} \frac{x}{z_j}. \tag{2.4.25}
\]

Substituting the value of \(T\) from equation \((2.4.11)\) in equation \((2.4.7)\), with the aid of

\[
s_4 = \sqrt{\frac{K_1}{K_3}}
\]

and the integrals \((2.4.22), (2.4.23)\), we obtain

\[
A_4 I_4 = \frac{H}{P_{34} \sqrt{K_3 / K_1}}, \tag{2.4.26}
\]
where

\[ I_4 = \left[ \tan^{-1} \frac{x}{s_4 \bar{a}} \right]_{x=\bar{b}}^{x=-\bar{b}} + \left[ \tan^{-1} \frac{\bar{b}}{z_4} \right]_{z=0}^{z=\bar{a}} = -\pi. \]  

(2.4.27)

\[ A_4 \] can be determined from equations (2.4.26) and (2.4.27) as follows

\[ A_4 = -\frac{H}{\pi P_{34}{\sqrt{K_3 / K_4}}}. \]  

(2.4.28)

Substituting the value of \( C \) from equation (2.4.12) in equation (2.4.8) and with the aid of the integrals (2.4.24), (2.4.25), we obtain

\[ \sum_{j=1}^{4} A_j P_{2j} r_j = 0, \]  

(2.4.29)

where

\[ r_j = \left[ s_j^2 \tan^{-1} \frac{x}{s_j \bar{a}} \right]_{x=\bar{b}}^{x=-\bar{b}} - \left[ \tan^{-1} \frac{\bar{b}}{s_j z} - \tan^{-1} \left( \frac{-\bar{b}}{s_j z} \right) \right]_{z=0}^{z=\bar{a}}, \]

i.e.,

\[ r_j = 2(s_j^2 - 1) \tan^{-1} \frac{\bar{b}}{s_j \bar{a}} + \pi. \]

The constants \( A_j (j = 1, 2, 3) \) are determined from equations (2.4.16), (2.4.17) and (2.4.29) by using the method of Cramer’s rule.
2.5 Green’s functions for a steady line heat source in a semi-infinite orthotropic thermoelastic diffusion material

We consider a semi infinite orthotropic thermoelastic diffusion material $z \geq 0$. A point heat source of strength $H$ is applied at the point $(0, h)$ in two dimensional Cartesian coordinate $(x, z)$ and the surface $z = 0$ has been taken stress free, thermally insulated and impermeable boundary. The geometry of the problem is shown in Figure 2.2. The general solution given by equations (2.3.26) and (2.3.34) is derived in this section.
**Figure 2.1** Geometry of the problem

**Figure 2.2** Geometry of the problem
For future reference, following notations are introduced

\[ z_j = s_j z, \quad h_k = s_k h, \quad z_{jk} = z_j + h_k, \]

\[ r_{jk} = \sqrt{x^2 + z_{jk}^2}, \quad \bar{z}_{jk} = z_j - h_k, \quad \bar{r}_{jk} = \sqrt{x^2 + \bar{z}_{jk}^2}, \quad (j, k = 1, 2, 3, 4). \]

(2.5.1)

Following Hou et al. (2009), Green’s functions in the semi-infinite plane are assumed of the following form:

\[
\psi_j = A_j \left[ \frac{1}{2} (\bar{z}_{jj} - x^2) \left( \log \bar{r}_{jj} - \frac{3}{2} \right) - x \bar{z}_{jj} \tan^{-1} \left( \frac{x}{\bar{z}_{jj}} \right) \right] + \sum_{k=1}^{4} A_{jk} \left[ \frac{1}{2} (\bar{z}_{jk} - x^2) \left( \log r_{jk} - \frac{3}{2} \right) - x \bar{z}_{jk} \tan^{-1} \left( \frac{x}{\bar{z}_{jk}} \right) \right], \tag{2.5.2}
\]

where \( A_j \) and \( A_{jk} (j, k = 1, 2, 3, 4) \) are twenty constant to be determined.

Substituting the value \( \psi_j (j = 1, 2, 3, 4) \) from equation (2.5.2) in equations (2.3.26) and (2.3.34) yield the expressions for displacement components, temperature change, mass concentration, and stress components as follows

\[
u = -\sum_{j=1}^{4} A_j \left[ x (\log r_{jj} - 1) + \bar{z}_{jj} \tan^{-1} \frac{x}{\bar{z}_{jj}} \right]
- \sum_{j=1}^{4} \sum_{k=1}^{4} A_{jk} \left[ x (\log r_{jk} - 1) + \bar{z}_{jk} \tan^{-1} \frac{x}{\bar{z}_{jk}} \right], \tag{2.5.3}
\]

\[
w = \sum_{j=1}^{4} s_j P_{1j} A_j \left[ \bar{z}_{jj} (\log \bar{r}_{jj} - 1) - x \tan^{-1} \frac{x}{\bar{z}_{jj}} \right]
+ \sum_{j=1}^{4} \sum_{k=1}^{4} s_j P_{1j} A_{jk} \left[ \bar{z}_{jk} (\log \bar{r}_{jk} - 1) - x \tan^{-1} \frac{x}{\bar{z}_{jk}} \right]. \tag{2.5.4}
\]
\[ T = P_{34}A_4 \log r_{44} + P_{34} \sum_{k=1}^{4} A_{4k} \log r_{4k}, \]  
(2.5.5)

\[ C = \sum_{j=1}^{4} P_{2j}A_j \log \bar{r}_{jj} + \sum_{j=1}^{4} \sum_{k=1}^{4} P_{2jk}A_{jk} \log r_{jk}, \]  
(2.5.6)

\[ \sigma_{xx} = -\sum_{j=1}^{4} s_j^2 w_{1j}A_j \log \bar{r}_{jj} - \sum_{j=1}^{4} \sum_{k=1}^{4} s_j^2 w_{1j}A_{jk} \log r_{jk}, \]  
(2.5.7)

\[ \sigma_{zz} = \sum_{j=1}^{4} w_{1j}A_j \log \bar{r}_{jj} + \sum_{j=1}^{4} \sum_{k=1}^{4} w_{1j}A_{jk} \log r_{jk}, \]  
(2.5.8)

\[ \sigma_{zx} = \sum_{j=1}^{4} s_j w_{1j}A_j \tan^{-1} \frac{x}{z_{jj}} - \sum_{j=1}^{4} \sum_{k=1}^{4} s_j w_{1j}A_{jk} \tan^{-1} \frac{x}{z_{jk}}. \]  
(2.5.9)

Considering the continuity at the plane \( z = h \) for \( w \) and \( \sigma_{zr} \) yield the following expressions

\[ \sum_{j=1}^{4} s_j P_{1j}A_j = 0, \]  
(2.5.10)

\[ \sum_{j=1}^{4} s_j w_{1j}A_j = 0. \]  
(2.5.11)

Substituting the value of \( w_{1j} \) from equation (2.3.35) in equation (2.5.11) yield

\[ \sum_{j=1}^{4} s_j h_4 (1 + P_{1j})A_j = 0. \]  
(2.5.12)

By virtue of equation (2.5.10), equation (2.5.12) can be simplified as

\[ \sum_{j=1}^{4} s_j A_j = 0. \]  
(2.5.13)
When the mechanical, thermal and concentration equilibrium for a rectangle of $0 < \tilde{a}_1 < h < \tilde{a}_2$ ($0 < \mathbf{b} \leq x \leq \mathbf{b}$) are considered (Figure 2.2) three equations can be obtained

\begin{align*}
\int_{-\mathbf{b}}^{\mathbf{b}} \left[ \sigma_{zz}(x, \tilde{a}_2) - \sigma_{zz}(x, \tilde{a}_1) \right] dx + \int_{\tilde{a}_1}^{\tilde{a}_2} \left[ \sigma_{zx}(\mathbf{b}, z) - \sigma_{zx}(-\mathbf{b}, z) \right] dz &= 0, \\
-\varepsilon_3 \int_{-\mathbf{b}}^{\mathbf{b}} \left[ \frac{\partial T}{\partial z}(x, \tilde{a}_2) - \frac{\partial T}{\partial z}(x, \tilde{a}_1) \right] dx - \int_{\tilde{a}_1}^{\tilde{a}_2} \left[ \frac{\partial T}{\partial x}(\mathbf{b}, z) - \frac{\partial T}{\partial x}(-\mathbf{b}, z) \right] dz &= H, \\
\int_{-\mathbf{b}}^{\mathbf{b}} \left[ \frac{\partial C}{\partial z}(x, \tilde{a}_2) - \frac{\partial C}{\partial z}(x, \tilde{a}_1) \right] dx + \int_{\tilde{a}_1}^{\tilde{a}_2} \left[ \frac{\partial C}{\partial x}(\mathbf{b}, z) - \frac{\partial C}{\partial x}(-\mathbf{b}, z) \right] dz &= 0.
\end{align*}

Some useful integrals are given as follows

\begin{align*}
\int \log \tilde{r}_{jj} dx &= x(\log \tilde{r}_{jj} - 1) + \tilde{z}_{jj} \tan^{-1}\left( \frac{x}{\tilde{z}_{jj}} \right), \\
\int \log r_{jk} dx &= x(\log r_{jk} - 1) + \tilde{z}_{jk} \tan^{-1}\left( \frac{x}{\tilde{z}_{jk}} \right), \\
\int \frac{\partial T}{\partial z} dx &= s_4 P_{34} \left( A_4 \tan^{-1}\frac{x}{\tilde{z}_{44}} + \sum_{k=1}^{4} A_{4k} \tan^{-1}\frac{x}{\tilde{z}_{4k}} \right), \\
\int \frac{\partial T}{\partial x} dz &= -\frac{P_{34}}{s_4} \left( A_4 \tan^{-1}\frac{x}{\tilde{z}_{44}} + \sum_{k=1}^{4} A_{4} \tan^{-1}\frac{x}{\tilde{z}_{4k}} \right), \\
\int \frac{\partial C}{\partial z} dx &= A_{j} s_{j} P_{2j} \tan^{-1}\frac{x}{\tilde{z}_{jj}} + \sum_{k=1}^{4} A_{jk} s_{j} P_{2j} \tan^{-1}\frac{x}{\tilde{z}_{jk}}, \\
\int \frac{\partial C}{\partial x} dz &= -\frac{A_{j}}{s_{j}} P_{2j} \tan^{-1}\frac{x}{\tilde{z}_{jj}} - \sum_{k=1}^{4} \frac{A_{jk}}{s_{j}} P_{2j} \tan^{-1}\frac{x}{\tilde{z}_{jk}}.
\end{align*}
It is noticed that the integrals (2.5.20) and (2.5.22) are not continuous at $z = h$, thus the following expression should be used

$$\int_{\bar{a}_1}^{a} \frac{\partial T}{\partial \bar{z}} \, dz = \int_{\bar{a}_1}^{a} \frac{\partial T}{\partial x} \, dz + \int_{h^+}^{a} \frac{\partial T}{\partial \bar{z}} \, dz, \tag{2.5.23}$$

$$\int_{\bar{a}_1}^{a} \frac{\partial C}{\partial \bar{z}} \, dz = \int_{\bar{a}_1}^{a} \frac{\partial C}{\partial x} \, dz + \int_{h^+}^{a} \frac{\partial C}{\partial \bar{z}} \, dz. \tag{2.5.24}$$

Making use the values of $\sigma_{zz}$ and $\sigma_{zx}$ from equations (2.5.8) and (2.5.9) in equation (2.5.14) and using the integrals (2.5.17), (2.5.18), we obtain

$$\sum_{j=1}^{4} w_{1j} A_{j1} + \sum_{j=1}^{4} w_{1j} \sum_{k=1}^{4} A_{jk} I_2 = 0, \tag{2.5.25}$$

where

$$I_1 = \left[ \left( x(\log \bar{r}_{jj} - 1) + \bar{z}_{jj} \tan^{-1} \left( \frac{X}{\bar{z}_{jj}} \right) \right) \right]_{z=\bar{a}_2}^{x=b} \left[ \left( x \log \bar{r}_{jj} + \bar{z}_{jj} \tan^{-1} \left( \frac{X}{\bar{z}_{jj}} \right) \right) \right]_{z=\bar{a}_1}^{x=b} = 0, \tag{2.5.26}$$

$$I_2 = \left[ \left( x(\log r_{jk} - 1) + z_{jk} \tan^{-1} \left( \frac{X}{z_{jk}} \right) \right) \right]_{z=\bar{a}_2}^{x=b} \left[ \left( x \log r_{jk} + z_{jk} \tan^{-1} \left( \frac{X}{z_{jk}} \right) \right) \right]_{z=\bar{a}_1}^{x=b} = 0. \tag{2.5.27}$$

Equations (2.5.26) and (2.5.27) show that the equations (2.5.14) and (2.5.25) are satisfied automatically.
Substituting the value of $T$ from equation (2.5.5) in equation (2.5.15), with the aid of $s_4 = \sqrt{K_1 / K_3}$ and integrals (2.5.19), (2.5.20) and (2.5.23), yield

$$A_4 I_5 + \sum_{k=1}^{4} A_{4k} I_6 = \frac{H}{P_{34} \sqrt{K_3 / K_1}},$$

(2.5.28)

where

$$I_5 = \left[ \left( \tan^{-1} \left( \frac{x}{z_{44}} \right) \right)_{x=\bar{b}} \right]^{x=\bar{b}}_{z=a_2} - \left[ \left( \tan^{-1} \left( \frac{x}{z_{44}} \right) \right)_{x=-\bar{b}} \right]^{z=h}_{z=a_1} + \left[ \left( \tan^{-1} \left( \frac{x}{z_{44}} \right) \right)_{x=\bar{b}} \right]^{z=a_2}_{z=h},$$

(2.5.29)

$$I_6 = \left[ \left( \tan^{-1} \left( \frac{x}{z_{4k}} \right) \right)_{x=\bar{b}} \right]^{z=a_2}_{z=\bar{b}} - \left[ \left( \tan^{-1} \left( \frac{x}{z_{4k}} \right) \right)_{z=a_1} \right]^{x=\bar{b}} = 0.$$  

(2.5.30)

From equations (2.5.28), (2.5.29) and (2.5.30), we obtain

$$A_4 = -\frac{H}{2\pi P_{34} \sqrt{K_3 / K_1}}.$$  

(2.5.31)

Substituting the value of $C$ from equation (2.5.6) in equation (2.5.16) and using the integrals (2.5.21), (2.5.22) and (2.5.24), we obtain

$$\sum_{j=1}^{4} P_{2j} A_{r_j} + \sum_{j=1}^{4} r_j \sum_{k=1}^{4} P_{2j} A_{jk} = 0,$$

(2.5.32)

where
\[ r_j = \left[ s_j^2 \tan^{-1} \left( \frac{x}{z_{jj}} \right) \right]_{z=\bar{a}_1}^{x=\bar{b}} - \left[ \tan^{-1} \left( \frac{x}{z_{jj}} \right) \right]_{x=-\bar{b}}^{z=\bar{a}_1} + \left[ \tan^{-1} \left( \frac{x}{z_{jj}} \right) \right]_{x=-\bar{b}}^{z=\bar{a}_2} + 2(s_j^2 - 1) \left[ \tan^{-1} \left( \frac{\bar{b}}{s_j\bar{a}_2 - s_jh} \right) - \tan^{-1} \left( \frac{\bar{b}}{s_j\bar{a}_1 - s_jh} \right) \right] + 2\pi \]

and

\[ r_j^* = \left[ s_j^2 \left( \tan^{-1} \left( \frac{x}{z_{jj}} \right) \right) \right]_{z=\bar{a}_1}^{x=\bar{b}} - \left[ \tan^{-1} \left( \frac{x}{z_{jj}} \right) \right]_{x=-\bar{b}}^{z=\bar{a}_1} = 2(s_j^2 - 1) \left[ \tan^{-1} \left( \frac{\bar{b}}{s_j\bar{a}_2 + s_jh} \right) - \tan^{-1} \left( \frac{\bar{b}}{s_j\bar{a}_1 + s_jh} \right) \right] \]

Equation (2.5.1) at the surface \( z = 0 \) gives

\[ z_j = 0, \quad h_k = s_kh, \quad z_{jk} = h_k, \]

\[ r_{jk} = \sqrt{x^2 + h_k^2}, \quad \bar{z}_{jk} = -h_k, \quad \bar{r}_{jk} = \sqrt{x^2 + h_k^2} \]  \hspace{1cm} (2.5.33)

Substituting the values of \( \sigma_{zz}, \sigma_{zx}, T \) and \( C \) from equations (2.5.5), (2.5.6), (2.5.8) and (2.5.9) in equations (2.4.3)-(2.4.5), with the aid of \( s_4 = \sqrt{K_1/K_3} \) and equation (2.5.33), yield

\[ -s_jw_{1j}A_j + \sum_{k=1}^{4} s_k w_{1k} A_{kj} = 0, \quad j = 1,2,3,4 \]  \hspace{1cm} (2.5.34)

\[ w_{1j}A_j + \sum_{k=1}^{4} w_{1k} A_{kj} = 0, \quad j = 1,2,3,4 \]  \hspace{1cm} (2.5.35)
\[A_4 - A_{44} = 0, \quad A_{4k} = 0, \quad k = 1, 2, 3\]  \hspace{1cm} (2.5.36)

\[P_{2j} A_j - \sum_{k=1}^{4} P_{2k} A_{kj} \cdot \quad j = 1, 2, 3, 4\]  \hspace{1cm} (2.5.37)

We can determine the nineteen constants \( A_j (j = 1, 2, 3) \) and \( A_{jk} (j, k = 1, 2, 3, 4) \) from nineteen equations including equations (2.5.10), (2.5.13), (2.5.32), (2.5.34), (2.5.35), (2.5.36) and (2.5.37) by the method of Cramer’s rule.

### 2.6 Special cases

#### 2.6.1 Case I

In the absence of diffusion effects i.e. \( b_1 = b_3 = a = b = 0 \) in equations (2.4.9)-(2.4.15) yield

\[u = -\sum_{j=1}^{3} A_j \left[ x (\log r_j - 1) + z_j \tan^{-1} \frac{x}{z_j} \right], \quad (2.6.1.1)\]

\[w = \sum_{j=1}^{3} s_j \bar{P}_{1j} A_j \left[ z_j (\log r_j - 1) - x \tan^{-1} \frac{x}{z_j} \right], \quad (2.6.1.2)\]

\[T = A_3 \bar{P}_{23} \log r_3, \quad (2.6.1.3)\]

\[\sigma_{xx} = -\sum_{j=1}^{3} s_j^2 w_{1j} A_j \log r_j, \quad (2.6.1.4)\]

\[\sigma_{zz} = \sum_{j=1}^{3} w_{1j} A_j \log r_j, \quad (2.6.1.5)\]

\[\sigma_{zx} = -\sum_{j=1}^{4} s_j w_{1j} A_j \tan^{-1} \frac{x}{z_j}. \quad (2.6.1.6)\]
where \( s_1, s_2, s_3, s_4 \) in this case are reduces to \( s_1, s_2, s_3 \) with \( s_3 = \sqrt{\frac{K_1}{K_3}} \) and \( s_1, s_2 \) are two roots (with positive real part) of the equation

\[
\varsigma_1 s^4 - \varsigma_2 s^2 + \varsigma_3 = 0,
\]

(2.6.1.7)

where

\[
\varsigma_1 = \delta_1 \delta_3, \quad \varsigma_2 = \tilde{\delta}_3 + \delta_1^2 - \tilde{\delta}_2^2, \quad \varsigma_3 = \delta_1.
\]

\[
\bar{p}_{1j} = \frac{\bar{p}_{2j}}{\bar{p}_{1j}}, \quad \bar{p}_{23} = \frac{\bar{p}_{33}}{\bar{p}_{13}},
\]

\[
\bar{p}_{kj} = -u_k + \bar{u}_k s_j^2, \quad (j, k = 1, 2)
\]

\[
\bar{p}_{33} = \varsigma_1 s_3^4 - \varsigma_2 s_3^2 + \varsigma_3,
\]

\[
u_1 = \delta_1, \quad \nu_2 = \epsilon_1 - \delta_2, \quad \bar{\nu}_1 = \delta_3 - \epsilon_1 \delta_2, \quad \bar{\nu}_2 = -\delta_1 \epsilon_2.
\]

Also the boundary conditions in this case are given by equations (2.4.3) and (2.4.4) with the change values of \( \sigma_{zz}, \sigma_{zx} \) and \( T \).

Substituting the values of \( \sigma_{zz}, \sigma_{zx} \) and \( T \) from equations (2.6.1.3), (2.6.1.5) and (2.6.1.6) in equations (2.4.3)-(2.4.4), we obtain

\[
\sum_{j=1}^{3} w_{1j} A_j = 0, \quad \text{(2.6.1.8)}
\]

\[
\sum_{j=1}^{3} s_j w_{1j} A_j = 0, \quad \text{(2.6.1.9)}
\]

\[
zs_3^2 \bar{p}_{23} A_3 = 0. \quad \text{(2.6.1.10)}
\]

Equation (2.6.1.10) is automatically satisfied at the surface \( (z = 0) \).
Making use the values of $\sigma_{zz}$, $\sigma_{xx}$ and $T$ from equations (2.6.1.3), (2.6.1.5) and (2.6.1.6) in equations (2.4.6)-(2.4.7) and with the aid of $s_3 = \sqrt{\frac{K_1}{K_3}}$, we obtain the two equations, one is similar to the equation (2.6.1.8) and other is given by

$$A_3 = -\frac{H}{2\pi P_{23}\sqrt{K_3/K_1}}.$$  

(2.6.1.11)

The constants $A_j (j = 1,2)$ are determined by two equations (2.6.1.8) and (2.6.1.9).

The above results are similar as obtained by Hou et al. (2011).

### 2.6.2 Case II

In the absence of diffusion effects i.e. $b_1 = b_3 = a = b = 0$, in equations (2.5.3)-(2.5.9) yield

$$u = \sum_{j=1}^{3} A_j \left[ x(\log r_{jj} - 1) + z_{jj} \tan^{-1} \frac{x}{z_{jj}} \right] - \sum_{j=1}^{3} \sum_{k=1}^{3} A_{jk} \left[ x(\log r_{jk} - 1) + z_{jk} \tan^{-1} \frac{x}{z_{jk}} \right],$$  

(2.6.2.1)

$$w = -\sum_{j=1}^{3} s_j \tilde{P}_{1j} A_j \left[ z_{jj} (\log r_{jj} - 1) - x \tan^{-1} \frac{x}{z_{jj}} \right] + \sum_{j=1}^{3} \sum_{k=1}^{3} s_j \tilde{P}_{1j} A_{jk} \left[ z_{jk} (\log r_{jk} - 1) - x \tan^{-1} \frac{x}{z_{jk}} \right],$$  

(2.6.2.2)

$$T = \tilde{P}_{23} A_4 \log r_{44} + \tilde{P}_{34} \sum_{k=1}^{3} A_{2k} \log r_{2k},$$  

(2.6.2.3)

$$\sigma_{xx} = -\sum_{j=1}^{3} s_j^2 w_{1j} A_j \log r_{jj} - \sum_{j=1}^{3} \sum_{k=1}^{3} s_j^2 w_{1j} A_{jk} \log r_{jk},$$  

(2.6.2.4)
\[ \sigma_{zz} = \sum_{j=1}^{3} w_{1j} A_{j} \log \tilde{r}_{jj} + \sum_{j=1}^{3} \sum_{k=1}^{3} w_{1j} A_{jk} \log r_{jk}, \]  
\[ (2.6.2.5) \]

\[ \sigma_{zx} = -\sum_{j=1}^{3} s_{j} w_{1j} A_{j} \tan^{-1} \frac{x}{z_{jj}} - \sum_{j=1}^{3} \sum_{k=1}^{3} s_{j} w_{1j} A_{jk} \tan^{-1} \frac{x}{z_{jk}}, \]  
\[ (2.6.2.6) \]

where \( s_{j}(j=1,2,3) \) are same as given in case I

Consider the continuity of plane of \( z = h \) for \( w \) and \( \sigma_{zx} \), and using the equations (2.6.2.3), (2.6.2.5) and (2.6.2.6) in equations (2.4.3) and (2.4.4), with the change value of \( \sigma_{zz}, \sigma_{zz} \), \( T \) and with the help of \( s_{3} = \sqrt{K_{1}/K_{3}}, \quad A_{3} = \frac{H}{2\pi P_{23} \sqrt{K_{3}/K_{1}}} \) and equation (2.5.33) yield the following expressions in the absence of diffusion

\[ \sum_{j=1}^{3} s_{j} \tilde{P}_{1j} A_{j} = 0, \]  
\[ (2.6.2.7) \]

\[ \sum_{j=1}^{3} s_{j} A_{j} = 0, \]  
\[ (2.6.2.8) \]

\[- s_{j} w_{j} A_{j} + \sum_{k=1}^{3} s_{k} w_{k} A_{kj} = 0, \quad j = 1,2,3 \]  
\[ (2.6.2.9) \]

\[ w_{j} A_{j} + \sum_{k=1}^{3} w_{k} A_{kj} = 0, \quad j = 1,2,3 \]  
\[ (2.6.2.10) \]

\[ A_{3} - A_{33} = 0, \quad A_{jk} = 0, \quad k = 1,2. \]  
\[ (2.6.2.11) \]

The eleven constants \( A_{j}(j=1,2) \) and \( A_{jk}(j,k=1,2,3) \) are determined from eleven equations, (2.6.2.7)-(2.6.2.11) by the method of Cramer’s rule.

The above results are similar as obtained by Hou et al. (2009).
2.7 Numerical results and discussion

For numerical computations, we take the following values of the relevant parameter for orthotropic thermoelastic diffusion material as

\[ c_{11} = 18.78 \times 10^{10} \text{Kg.m}^{-1}\text{s}^{-2}, \quad c_{13} = 8.0 \times 10^{10} \text{Kg.m}^{-1}\text{s}^{-2}, \quad c_{33} = 10.2 \times 10^{10} \text{Kg.m}^{-1}\text{s}^{-2}, \]
\[ c_{55} = 10.06 \times 10^{10} \text{Kg.m}^{-1}\text{s}^{-2}, \quad T_0 = 0.293 \times 10^3 \text{K}, \quad \alpha_0 = 1.96 \times 10^{-5} \text{K}^{-1}, \quad \alpha_3 = 1.4 \times 10^{-5} \text{K}^{-1}, \]
\[ \alpha_1 = 1.1 \times 10^{-4} \text{m}^3.\text{Kg}^{-1}, \quad \alpha_3 = 0.43 \times 10^{-4} \text{m}^3.\text{Kg}^{-1}, \quad \alpha_4 = 1.1 \times 10^3 \text{W.m}^{-1}\text{K}^{-1}, \]
\[ K_3 = 0.33 \times 10^3 \text{W.m}^{-1}\text{K}^{-1}, \quad a = 1.4 \times 10^4 \text{m}^2\text{s}^{-2}.\text{K}^{-1}, \quad b = 9 \times 10^2 \text{Kg.m}^5\text{s}^{-2}, \]
\[ \alpha_1 = 0.58 \times 10^{-8} \text{m}^{-3}\text{s}\text{Kg}, \quad \alpha_3 = 0.52 \times 10^{-8} \text{m}^{-3}\text{s}\text{Kg} \]

\[ \alpha_1 = 0.58 \times 10^{-8} \text{m}^{-3}\text{s}\text{Kg}, \quad \alpha_3 = 0.52 \times 10^{-8} \text{m}^{-3}\text{s}\text{Kg} \]

and

\[ \alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha = 1, \quad \alpha = 1. \]

Figures 2.3 - 2.6 depict the variations of components of displacement, temperature change and mass concentration for a line heat source applied at the origin and Figs. 2.7-2.10 exhibit the variations of components of stress, temperature change and mass concentration for a line heat source applied at the point \((0, h)\). The solid line and dotted line are, respectively, correspond to thermoelastic (WTD \(z = 5\)) and (WTD \(z = 10\)) and center symbol on these lines are, respectively, correspond to thermoelastic diffusion (WD \(z = 5\)) and (WD \(z = 10\)).

Figure 2.3 depicts the variation of horizontal displacement \(u\) with \(x\) and it indicates that the values of \(u\) increase slightly for smaller values of \(x\), but for higher values of \(x\), the values of \(u\) increase monotonically. It is noticed that the values of \(u\) in case of WTD remain more in comparison with WD. Figure 2.4 depicts the variation of vertical displacement \(w\) with \(x\). It is evident that the values of \(w\) decrease monotonically for the case of WTD, whereas for the case of WD \((z = 5)\) the values of \(w\) decrease for smaller values of \(x\) and increase for higher values of \(x\), but in case WD \((z = 10)\) reverse behavior
occurs. It is evident that the values of \( w \) in case of WTD \((z = 10)\) remain more in comparison with other cases.

Figure 2.5 exhibits the variation of temperature change \( T \) with \( x \) and it indicates that the values of \( T \) in case of WD increases slightly, although for the case of WTD the values of \( T \) increase monotonically. It is noticed that the values of \( T \) in case of WTD remain more in comparison with the case WD for higher values of \( x \). Figure 2.6 depicts the variation of mass concentration \( C \) with \( x \) and it represents that for all values of \( x \), the values of \( C \) decrease in case of WD \((z = 10)\), whereas for the case of WD \((z = 5)\) the values of \( C \) decrease for smaller values of \( x \), but for higher values of \( x \) reverse behavior occurs.

Figure 2.7 exhibits the variation of normal stress component \( \sigma_{zz} \) with \( x \) and it indicates that the values of \( \sigma_{zz} \) increase for the case of WTD, but for WD it remains oscillatory. It is evident that for \( z = 5 \), the values of \( \sigma_{zz} \) remain more as \( x \) varies (i) \( 0 \leq x \leq 3.70 \) for WD (ii) \( 3.70 < x \leq 4.2 \) for WTD (iii) \( x > 4.2 \) for WD, in comparison with each other case. Figure 2.8 displays the variation of tangential stress component \( \sigma_{zx} \) with \( x \) and it is noticed that in the case of WTD, the values of \( \sigma_{zx} \) decrease, but for WD, the values of \( \sigma_{zx} \) decrease when \( 0 \leq x \leq 3 \) and increases further (i.e. for \( x > 3 \)). It is also noticed that the values of \( \sigma_{zx} \) remain more for WTD, when \( 0 \leq x \leq 3.5 \) in comparison with WD, but for \( x > 3.5 \), the values of WD remain more in comparison with WTD.

Figure 2.9 depicts the variation of temperature change \( T \) with \( x \) and it indicates that when \( 0 \leq x \leq 4 \), the values of \( T \) slightly increase and for \( x > 4 \), the values of \( T \) increase monotonically for both cases (WD, WTD). It is noticed that the values of \( T \) in case of WTD remain more in comparison with WD. Figure 2.10 shows the variation of mass concentration \( C \) with \( x \) and it is evident that the values of \( C \) decrease with oscillation when \( 0 \leq x \leq 4 \) and for \( x > 4 \), the values of \( C \) increase for both cases (WD, WTD) whereas for \( z = 5 \), the values of \( C \) remain more when \( 0 \leq x \leq 3.5 \) (In comparison with \( z = 10 \)), but for \( 3.5 \leq x \leq 4.5 \), the values of \( C \) remain more for \( z = 10 \) in comparison with \( z = 5 \).
2.8 Conclusion

In the present chapter, the fundamental solution and Green’s functions for two dimensional problem in orthotropic thermoelastic diffusion media are derived. After applying the dimensionless quantities and using the operator theory, we have derived the general expression for components of displacement, stress, temperature change and mass concentration. On the basis of general solution, the fundamental solution and Green’s functions for a line heat source in semi-infinite orthotropic thermoelastic diffusion media has been derived. The components of displacement, stress, temperature change and mass concentration are expressed in terms of elementary functions. Since all the components are expressed in terms of elementary functions, it is convenient to use. Some special cases are also deduced.

From the numerical and graphically results, it is observed that when point heat source is applied on the surface of a semi-infinite, the values of horizontal displacement $u$, vertical displacement $w$ and temperature change $T$ in case of WTD remain more as comparison with the case of WD. When point source is applied in the interior of a semi-infinite thermoelastic diffusion material, the values of normal stress component $\sigma_{zz}$ and temperature change $T$ remain more in case of WTD (in comparison with WD) for higher values of $x$, whereas the values of normal stress component $\sigma_{zx}$ remain more in case of WD in comparison with WTD.
Figure 2.3 Variation of horizontal displacement $u$ with respect to $x$.

Figure 2.4 Variation of vertical displacement $w$ with respect to $x$. 
Figure 2.5 Variation of temperature change $T$ with respect to $x$

Figure 2.6 Variation of mass concentration $C$ with respect to $x$
Figure 2.7 Variation of normal stress $\sigma_{zz}$ with respect to $x$

Figure 2.8 Variation of tangential stress $\sigma_{zx}$ with respect to $x$
Figure 2.9 Variation of temperature change $T$ with respect to $x$

Figure 2.10 Variation of mass concentration $C$ with respect to $x$