CHAPTER – 5

$\omega$-EXTREMALLY DISCONNECTEDNESS IN L-FUZZY TOPOLOGICAL SPACES

A new class of fuzzy topological spaces called L-fuzzy $\omega$-extremally disconnected spaces is introduced in this chapter by using the concepts of L-fuzzy topological space [76], L-fuzzy $\omega$-closed set [72] and L-fuzzy extremally disconnected spaces [76]. Some interesting properties and characterizations are obtained. Tietze extension theorem for L-fuzzy $\omega$-extremally disconnected spaces has been discussed as in [76], besides proving several other propositions.
5.1 PROPERTIES OF L-FUZZY $\omega$-CONTINUOUS MAPPINGS AND L-FUZZY $\omega$-EXTREMALLY DISCONNECTED SPACES

In this section, the concepts of L-fuzzy $\omega$-continuous mappings and L-fuzzy $\omega$-extremally disconnected spaces are introduced. Some interesting properties of the concepts introduced are investigated.

Definition 5.1.1

Let $(X, T)$ be an L-fuzzy topological space. For any L-fuzzy set $\lambda$ in $(X, T)$, L-fuzzy $\omega$-closure of $\lambda$ (briefly, $L\omega$-cl ($\lambda$)) is defined as

$L\omega$-cl ($\lambda$) = $\land \{ \mu : \mu \geq \lambda$ and $\mu$ is L-fuzzy $\omega$-closed $\}.$

Definition 5.1.2

Let $(X, T)$ be an L-fuzzy topological space. For any L-fuzzy set $\lambda$ in $(X, T)$, L-fuzzy $\omega$-interior of $\lambda$ (briefly, $L\omega$-int ($\lambda$)) is defined as

$L\omega$-int ($\lambda$) = $\lor \{ \mu : \mu \leq \lambda$ and $\mu$ is L-fuzzy $\omega$-open $\}.$

Remark 5.1.1

Let $(X, T)$ be an L-fuzzy topological space. For any L-fuzzy set $\lambda$ in $(X, T)$, the following conditions hold:

(a) $1 - L\omega$-int ($\lambda$) = $L\omega$-cl ($1 - \lambda$).

(b) $1 - L\omega$-cl ($\lambda$) = $L\omega$-int ($1 - \lambda$).

Definition 5.1.3

Let $(X, T)$ and $(Y, S)$ be any two L-fuzzy topological spaces. A mapping $f : (X, T) \rightarrow (Y, S)$ is called L-fuzzy $\omega$-continuous if $f^{-1}(\lambda)$ is L-fuzzy $\omega$-closed in $(X, T)$, for every L-fuzzy closed set $\lambda$ in $(Y, S)$. 
Definition 5.1.4

Let \((X, T)\) and \((Y, S)\) be any two \(L\)-fuzzy topological spaces. A mapping \(f : (X, T) \to (Y, S)\) is called \(L\)-fuzzy \(\omega\)-irresolute if the inverse image of every \(L\)-fuzzy \(\omega\)-open set in \((Y, S)\) is \(L\)-fuzzy \(\omega\)-open in \((X, T)\).

Definition 5.1.5

Let \((X, T)\) and \((Y, S)\) be any two \(L\)-fuzzy topological spaces. A mapping \(f : (X, T) \to (Y, S)\) is said to be \(L\)-fuzzy \(\omega\)-open if the image of every \(L\)-fuzzy \(\omega\)-open set in \((X, T)\) is \(L\)-fuzzy \(\omega\)-open in \((Y, S)\).

Proposition 5.1.1

Let \((X, T)\) and \((Y, S)\) be any two \(L\)-fuzzy topological spaces. Then \(f : (X, T) \to (Y, S)\) is \(L\)-fuzzy \(\omega\)-irresolute iff \(f(L_\omega\text{-cl}(\lambda)) \leq L_\omega\text{-cl}(f(\lambda))\), for every \(L\)-fuzzy set \(\lambda\) in \((X, T)\).

Proof

Suppose \(f\) is \(L\)-fuzzy \(\omega\)-irresolute and \(\lambda\) be any \(L\)-fuzzy set in \((X, T)\). Then, \(L_\omega\text{-cl}(f(\lambda))\) is \(L\)-fuzzy \(\omega\)-closed in \((Y, S)\). By hypothesis, \(f^{-1}(L_\omega\text{-cl}(f(\lambda)))\) is \(L\)-fuzzy \(\omega\)-closed in \((X, T)\). And

\[
\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(L_\omega\text{-cl}(f(\lambda))).
\]

Hence by the definition of \(L\)-fuzzy \(\omega\)-closure,

\[
L_\omega\text{-cl}(\lambda) \leq f^{-1}(L_\omega\text{-cl}(f(\lambda))).
\]

That is,

\[
f(L_\omega\text{-cl}(\lambda)) \leq L_\omega\text{-cl}(f(\lambda)).
\]

Conversely, suppose that \(\lambda\) is \(L\)-fuzzy \(\omega\)-closed in \((Y, S)\). Now by hypothesis, \(f(L_\omega\text{-cl}(f^{-1}(\lambda))) \leq L_\omega\text{-cl}(f(f^{-1}(\lambda)))\).

This implies,

\[
L_\omega\text{-cl}(f^{-1}(\lambda)) \leq f^{-1}(\lambda). \text{ So that } f^{-1}(\lambda) = L_\omega\text{-cl}(f^{-1}(\lambda)).
\]
That is, $f^{-1}(\lambda)$ is L-fuzzy $\omega$-closed and so $f$ is L-fuzzy $\omega$-irresolute.

**Proposition 5.1.2**

Let $(X, T)$ and $(Y, S)$ be any two L-fuzzy topological spaces and let $f : (X, T) \rightarrow (Y, S)$ be an L-fuzzy $\omega$-open surjective mapping. Then $f^{-1}(L_\omega\text{-cl}(\lambda)) \leq L_\omega\text{-cl}(f^{-1}(\lambda))$, for each L-fuzzy set $\lambda$ in $(Y, S)$.

**Proof**

Let $\mu$ be any L-fuzzy set in $(Y, S)$ and let $\mu = f^{-1}(1 - \lambda)$. Then, $L_\omega\text{-int}(f^{-1}(1 - \lambda)) = L_\omega\text{-int}(\mu)$ is L-fuzzy $\omega$-open in $(X, T)$. Now, $L_\omega\text{-int}(\mu) \leq \mu$. Hence,

$$f(L_\omega\text{-int}(\mu)) \leq f(\mu).$$

That is,

$$L_\omega\text{-int}(f(L_\omega\text{-int}(\mu))) \leq L_\omega\text{-int}(f(\mu)).$$

Since $f$ is L-fuzzy $\omega$-open, $f(L_\omega\text{-int}(\mu))$ is L-fuzzy $\omega$-open in $(Y, S)$.

Therefore,

$$f(L_\omega\text{-int}(\mu)) \leq L_\omega\text{-int}(f(\mu))$$

$$= L_\omega\text{-int}(1 - \lambda).$$

Hence,

$$L_\omega\text{-int}(f^{-1}(1 - \lambda)) = L_\omega\text{-int}(\mu)$$

$$\leq f^{-1}(L_\omega\text{-int}(1 - \lambda)).$$

Therefore,

$$1 - (L_\omega\text{-int}(f^{-1}(1 - \lambda))) = 1 - L_\omega\text{-int}(\mu)$$

$$\geq 1 - (f^{-1}(L_\omega\text{-int}(1 - \lambda))).$$

Hence, $f^{-1}(1 - (L_\omega\text{-int}(1 - \lambda))) \leq L_\omega\text{-cl}(1 - f^{-1}(1 - \lambda))$.

Therefore,

$$f^{-1}(L_\omega\text{-cl}(\lambda)) \leq L_\omega\text{-cl}(f^{-1}(\lambda)).$$
Definition 5.1.6

Let \((X, T)\) be an L-fuzzy topological space. \((X, T)\) is called \(\omega\)-extremally disconnected if the L-fuzzy \(\omega\)-closure of every L-fuzzy \(\omega\)-open set is L-fuzzy \(\omega\)-open.

Proposition 5.1.3

The image \((Y, S)\) of an L-fuzzy \(\omega\)-extremally disconnected space \((X, T)\) under L-fuzzy \(\omega\)-irresolute, L-fuzzy \(\omega\)-open and surjective mapping is also L-fuzzy \(\omega\)-extremally disconnected.

Proof

Let \(\lambda\) be any L-fuzzy \(\omega\)-open set in \((Y, S)\). Since \(f\) is L-fuzzy \(\omega\)-irresolute, \(f^{-1}(\lambda)\) is L-fuzzy \(\omega\)-open in \((X, T)\). Therefore by assumption on \((X, T)\), it follows that \(L_\omega\text{-cl}(f^{-1}(\lambda))\) is L-fuzzy \(\omega\)-open in \((X, T)\). As \(f\) is L-fuzzy \(\omega\)-open, \(f(L_\omega\text{-cl}(f^{-1}(\lambda)))\) is L-fuzzy \(\omega\)-open in \((Y, S)\). By Proposition 5.1.2, \(f^{-1}(L_\omega\text{-cl}(\lambda)) \leq L_\omega\text{-cl}(f^{-1}(\lambda))\) and hence,

\[
f^{-1}(L_\omega\text{-cl}(\lambda)) = L_\omega\text{-cl}(\lambda)
\]

\[
\leq f(L_\omega\text{-cl}(f^{-1}(\lambda)))
\]

\[
\leq L_\omega\text{-cl}(f(f^{-1}(\lambda)))
\]

\[
= L_\omega\text{-cl}(\lambda), \text{ by Proposition 5.1.1.}
\]

This implies,

\[
L_\omega\text{-cl}(\lambda) = f(L_\omega\text{-cl}(f^{-1}(\lambda))
\]

and therefore, \(L_\omega\text{-cl}(\lambda)\) is L-fuzzy \(\omega\)-open in \((Y, S)\), proving that \((Y, S)\) is L-fuzzy \(\omega\)-extremally disconnected.
**Proposition 5.1.4**

Let \( \{ (X_\alpha, T_\alpha) / \alpha \in \Delta \} \) be a family of disjoint L-fuzzy \( \omega \)-extremally disconnected spaces and let \((X, T)\) be their L-fuzzy topological sum. Then \((X, T)\) is L-fuzzy \( \omega \)-extremally disconnected.

**Proof**

Let \( \lambda \) be an L-fuzzy \( \omega \)-open set in \((X, T)\). Then, \( \lambda/X_\alpha \) is L-fuzzy \( \omega \)-open in \((X_\alpha, T_\alpha)\). Since \((X_\alpha, T_\alpha)\) is L-fuzzy \( \omega \)-extremally disconnected, \( L\omega-cl_{X_\alpha} (\lambda/X_\alpha)\) is L-fuzzy \( \omega \)-open in \((X_\alpha, T_\alpha)\). Now,

\[
L\omega-cl_X (\lambda)/X_\alpha = L\omega-cl_{X_\alpha} (\lambda/X_\alpha),
\]

which implies that \( L\omega-cl_X (\lambda) \) is L-fuzzy \( \omega \)-open in \((X, T)\). Therefore, \((X, T)\) is L-fuzzy \( \omega \)-extremally disconnected.

**5.2 CHARACTERIZATIONS OF L-FUZZY \( \omega \)-EXTREMALLY DISCONNECTED SPACES**

**Proposition 5.2.1**

For an L-fuzzy topological space \((X, T)\) the following conditions are equivalent:

(a) \((X, T)\) is an L-fuzzy \( \omega \)-extremally disconnected space.

(b) For each L-fuzzy \( \omega \)-closed set \( \lambda \), \( L\omega-int (\lambda) \) is L-fuzzy \( \omega \)-closed.

(c) For each L-fuzzy \( \omega \)-open set \( \lambda \),

\[
L\omega-cl (\lambda) + L\omega-cl (1 - L\omega-cl (\lambda)) = 1.
\]

(d) For every pair of L-fuzzy \( \omega \)-open sets \( \lambda \) and \( \mu \) with \( L\omega-cl (\lambda) + \mu = 1 \), we have \( L\omega-cl (\lambda) + L\omega-cl (\mu) = 1 \).
Proof

(a) ⇒ (b) Let \( \lambda \) be any L-fuzzy \( \omega \)-closed set. Then, \( 1 - \lambda \) is L-fuzzy \( \omega \)-open. Now, \( \text{L}\omega\text{-cl} (1 - \lambda) = 1 - \text{L}\omega\text{-int} (\lambda) \). By (a), \( \text{L}\omega\text{-cl} (1 - \lambda) \) is L-fuzzy \( \omega \)-open, which implies that \( \text{L}\omega\text{-int} (\lambda) \) is L-fuzzy \( \omega \)-closed.

(b) ⇒ (c) Let \( \lambda \) be any L-fuzzy \( \omega \)-open set. Then, \( 1 - \lambda \) is L-fuzzy \( \omega \)-closed. By (b), \( \text{L}\omega\text{-int} (1 - \lambda) \) is L-fuzzy \( \omega \)-closed. Now,

\[
\text{L}\omega\text{-cl} (\lambda) + \text{L}\omega\text{-cl} (1 - \text{L}\omega\text{-cl} (\lambda)) = \text{L}\omega\text{-cl} (\lambda) + \text{L}\omega\text{-cl} (\text{L}\omega\text{-int} (1 - \lambda)).
\]

(5.2.1)

Therefore by (5.2.1),

\[
\text{L}\omega\text{-cl} (\lambda) + \text{L}\omega\text{-cl} (1 - \text{L}\omega\text{-cl} (\lambda)) = \text{L}\omega\text{-cl} (\lambda) + \text{L}\omega\text{-int} (1 - \lambda)
= \text{L}\omega\text{-cl} (\lambda) + 1 - \text{L}\omega\text{-cl} (\lambda)
= 1.
\]

Therefore, \( \text{L}\omega\text{-cl} (\lambda) + \text{L}\omega\text{-cl} (1 - \text{L}\omega\text{-cl} (\lambda)) = 1 \).

(c) ⇒ (d) Let \( \lambda \) and \( \mu \) be L-fuzzy \( \omega \)-open sets with \( \text{L}\omega\text{-cl} (\lambda) + \mu = 1 \).

(5.2.2)

Then by (c), \( 1 = \text{L}\omega\text{-cl} (\lambda) + \text{L}\omega\text{-cl} (1 - \text{L}\omega\text{-cl} (\lambda)) \).

By (5.2.2), \( 1 - \text{L}\omega\text{-cl} (\lambda) = \mu \). Therefore, \( \text{L}\omega\text{-cl} (\lambda) + \text{L}\omega\text{-cl} (\mu) = 1 \).

(d) ⇒ (a) Let \( \lambda \) be any L-fuzzy \( \omega \)-open set. Put \( \mu = 1 - \text{L}\omega\text{-cl} (\lambda) \). Then clearly, \( \mu \) is L-fuzzy \( \omega \)-open and \( \text{L}\omega\text{-cl} (\lambda) + \mu = 1 \). Therefore by (d),

\[
\text{L}\omega\text{-cl} (\lambda) + \text{L}\omega\text{-cl} (\mu) = 1.
\]

This implies, \( \text{L}\omega\text{-cl} (\lambda) \) is L-fuzzy \( \omega \)-open and so \( (X, T) \) is L-fuzzy \( \omega \)-extremally disconnected.
**Proposition 5.2.2**

Let \((X, T)\) be an \(L\)-fuzzy topological space. Then \((X, T)\) is \(L\)-fuzzy \(\omega\)-extremally disconnected iff for all \(L\)-fuzzy \(\omega\)-open set \(\lambda\) and an \(L\)-fuzzy \(\omega\)-closed set \(\mu\) such that \(\lambda \leq \mu\), \(L\omega\)-cl \((\lambda)\) \(\leq L\omega\)-int \((\mu)\).

**Proof**

Let \((X, T)\) be \(L\)-fuzzy \(\omega\)-extremally disconnected. Let \(\lambda\) be \(L\)-fuzzy \(\omega\)-open and \(\mu\) be \(L\)-fuzzy \(\omega\)-closed with \(\lambda \leq \mu\). Then by (b) of Proposition 5.2.1, \(L\omega\)-int \((\mu)\) is \(L\)-fuzzy \(\omega\)-closed. Also, since \(\lambda\) is \(L\)-fuzzy \(\omega\)-open and \(\lambda \leq \mu\), it follows that \(\lambda \leq L\omega\)-int \((\mu)\). Again, since \(L\omega\)-int \((\mu)\) is \(L\)-fuzzy \(\omega\)-closed it follows that \(L\omega\)-cl \((\lambda)\) \(\leq L\omega\)-int \((\mu)\).

Conversely, let \(\mu\) be any \(L\)-fuzzy \(\omega\)-closed set. Then, \(L\omega\)-int \((\mu)\) is \(L\)-fuzzy \(\omega\)-open in \((X, T)\) and \(L\omega\)-int \((\mu)\) \(\leq \mu\). Therefore by assumption, \(L\omega\)-cl \((L\omega\)-int \((\mu)\)) \(\leq L\omega\)-int \((\mu)\). This implies that \(L\omega\)-int \((\mu)\) is \(L\)-fuzzy \(\omega\)-closed. Hence by (b) of Proposition 5.2.1, it follows that \((X, T)\) is \(L\)-fuzzy \(\omega\)-extremally disconnected.

**Remark 5.2.1**

Let \((X, T)\) be an \(L\)-fuzzy \(\omega\)-extremally disconnected space. Let \(\{\lambda_i, 1-\mu_i / i \in \mathbb{N}\}\) be a collection such that \(\lambda_i\)'s, are \(L\)-fuzzy \(\omega\)-open and \(\mu_i\)'s are \(L\)-fuzzy \(\omega\)-closed and let \(\lambda, \mu\) are \(L\)-fuzzy \(\omega\)-clopen sets respectively. If \(\lambda_i \leq \lambda \leq \mu_j\) and \(\lambda_i \leq \mu \leq \mu_j\) for all \(i, j \in \mathbb{N}\), then there exists an \(L\)-fuzzy \(\omega\)-clopen set \(\gamma\) such that \(L\omega\)-cl \((\lambda_i)\) \(\leq \gamma \leq L\omega\)-int \((\mu_j)\), for all \(i, j \in \mathbb{N}\).

**Proof**

By Proposition 5.2.2, \(L\omega\)-cl \((\lambda_i)\) \(\leq L\omega\)-cl \((\lambda)\) \(\land L\omega\)-int \((\mu)\) \(\leq L\omega\)-int \((\mu_j)\),
for all $i, j \in \mathbb{N}$. Therefore, $\gamma = L\omega\text{-cl} ( \lambda ) \land L\omega\text{-int} ( \mu )$ is an L-fuzzy $\omega$-clopen set satisfying the required conditions.

**Proposition 5.2.3**

Let $(X, T)$ be an L-fuzzy $\omega$-extremally disconnected space. Let \{\(\lambda_r\)\}_{r \in \mathbb{Q}} and \{\(\mu_r\)\}_{r \in \mathbb{Q}} be monotone increasing collections of L-fuzzy $\omega$-open sets and L-fuzzy $\omega$-closed sets of $(X, T)$ and suppose that $\lambda_{q_1} \leq \mu_{q_2}$ whenever $q_1 < q_2$ ($\mathbb{Q}$ is the set of all rational numbers). Then there exists a monotone increasing collection \{\(\gamma_r\)\}_{r \in \mathbb{Q}} of L-fuzzy $\omega$-clopen sets of $(X, T)$ such that $L\omega\text{-cl} ( \lambda_{q_1} ) \leq ( \gamma_{q_2} )$ and $\gamma_{q_1} \leq L\omega\text{-int} ( \mu_{q_2} )$ whenever $q_1 < q_2$.

**Proof**

Let us arrange into a sequence \{\(q_n\)\} of all rational numbers (without repetitions). For every $n \geq 2$, we shall define inductively a collection \{\(\gamma_{q_i}/ 1 \leq i \leq n \)\} of L-fuzzy $\omega$-clopen sets $(X, T)$ such that

\[ L\omega\text{-cl} ( \lambda_{q_i} ) \leq \gamma_{q_i} \text{ if } q < q_i, \quad \gamma_{q_i} \leq L\omega\text{-int} ( \mu_{q_i} ) \text{ if } q_i < q, \text{ for all } i < n \quad (S_n). \]

By Proposition 5.2.2, the countable collections \{\(L\omega\text{-cl} ( \lambda_{q_i} )\)\} and \{\(L\omega\text{-int} ( \mu_{q_i} )\)\} satisfy $L\omega\text{-cl} ( \lambda_{q_1} ) \leq L\omega\text{-int} ( \mu_{q_2} )$ if $q_1 < q_2$. By Remark 5.2.1, there exists an L-fuzzy $\omega$-clopen set $\delta_1$ such that $L\omega\text{-cl} ( \lambda_{q_1} ) \leq \delta_1 \leq L\omega\text{-int} ( \mu_{q_2} )$. Setting $\gamma_{q_1} = \delta_1$, we get $(S_2)$.

Define $\psi = \bigvee \{\gamma_{q_i}/ i < n, q_i < q_n\} \lor \lambda_{q_n}$ and $\phi = \bigwedge \{\gamma_{q_j}/ j < n, q_j > q_n\} \land \mu_{q_n}$. Then we have,

\[ L\omega\text{-cl} ( \gamma_{q_1} ) \leq L\omega\text{-cl} ( \psi ) \leq L\omega\text{-int} ( \gamma_{q_j} ) \]

and
\[ \text{L}_\omega \text{-cl} ( \gamma_{q_i} ) \leq \text{L}_\omega \text{-int} ( \phi ) \leq \text{L}_\omega \text{-int} ( \gamma_{q_j} ) \]

whenever \( q_i < q_n < q_j \) \((i, j < n)\) as well as
\[ \lambda_q \leq \text{L}_\omega \text{-cl} ( \psi ) \leq \mu_q \text{ and } \lambda_q \leq \text{L}_\omega \text{-int} ( \phi ) \leq \mu_{q'} \text{ whenever } q < q_n < q'. \]

This shows that the countable collections \{ \gamma_{q_i} / i < n, q_i < q_n \} \cup \{ \lambda_q / q < q_n \} and \{ \gamma_{q_j} / j < n, q_j > q_n \} \cup \{ \mu_q / q > q_n \} together with \( \psi \) and \( \phi \) fulfill all the conditions of Remark 5.2.1. Hence, there exists an L-fuzzy \( \omega \)-clopen set \( \delta_n \) such that

\[ \text{L}_\omega \text{-cl} ( \delta_n ) \leq \mu_q \text{ if } q_n < q, \lambda_q \leq \text{L}_\omega \text{-int} ( \delta_n ) \text{ if } q < q_n, \]

\[ \text{L}_\omega \text{-cl} ( \gamma_{q_i} ) \leq \text{L}_\omega \text{-int} ( \delta_n ) \text{ if } q_i < q_n, \text{L}_\omega \text{-cl} ( \delta_n ) \leq \text{L}_\omega \text{-int} ( \gamma_{q_j} ) \text{ if } q_n < q_j, \]

where \( 1 \leq i, j \leq n - 1 \). Now, setting \( \gamma_{q_n} = \delta_n \) we obtain the fuzzy sets \( \gamma_{q_1}, \gamma_{q_2}, \gamma_{q_3} \ldots \gamma_{q_n} \) that satisfy (S_{n+1}). Therefore, the collection \{ \gamma_{q_i} / i=1,2,.. \} has the required property.

**Definition 5.2.1**

Let \((X, T)\) be an L-fuzzy topological space. A mapping \( f : X \rightarrow R(L) \) is called lower( resp.upper ) L-fuzzy \( \omega \)-continuous if \( f^{-1}( R_t ) \) ( resp. \( f^{-1}( L_t ) \) ) is L-fuzzy \( \omega \)-open ( resp. L-fuzzy \( \omega \)-open / L-fuzzy \( \omega \)-closed ) for each \( t \in R \).

**Proposition 5.2.4**

Let \((X, T)\) be any L-fuzzy topological space; let \( \lambda \in L^X \) and let \( f : X \rightarrow R(L) \) be such that

\[ f(x)(t) = \begin{cases} 
1 & \text{if } t < 0 \\
\lambda(x) & \text{if } 0 \leq t \leq 1 \\
0 & \text{if } t > 1 
\end{cases} \]
for all \( x \in X \). Then \( f \) is lower ( resp. upper ) \( \omega \)-continuous iff \( \lambda \) is L-fuzzy \( \omega \)-open ( resp. L-fuzzy \( \omega \)-open / L-fuzzy \( \omega \)-closed ).

**Proof**

\[
f^{-1}(R_t) = \begin{cases} 
1 & \text{if } t < 0 \\
\lambda & \text{if } 0 \leq t < 1 \\
0 & \text{if } t \geq 1 
\end{cases}
\]

implies that \( f \) is lower L-fuzzy \( \omega \)-continuous iff \( \lambda \) is L-fuzzy \( \omega \)-open.

\[
f^{-1}(L'_t) = \begin{cases} 
1 & \text{if } t \leq 0 \\
\lambda & \text{if } 0 < t \leq 1 \\
0 & \text{if } t > 1 
\end{cases}
\]

implies that \( f \) is upper L-fuzzy \( \omega \)-continuous iff \( \lambda \) is L-fuzzy \( \omega \)-open / L-fuzzy \( \omega \)-closed.

**Proposition 5.2.5**

Let \((X, T)\) be an L-fuzzy topological space and let \( \lambda \in L^X \). Then \( \chi_\lambda \) is lower ( resp. upper ) L-fuzzy \( \omega \)-continuous iff \( \lambda \) is L-fuzzy \( \omega \)-open ( resp. L-fuzzy \( \omega \)-open / L-fuzzy \( \omega \)-closed ).

**Proof**

The proof follows from Proposition 5.2.4.

**Definition 5.2.2**

Let \((X, T)\) and \((Y, S)\) be any two L-fuzzy topological spaces. A mapping \( f : (X, T) \rightarrow (Y, S) \) is called strong L-fuzzy \( \omega \)-continuous if \( f^{-1}(\lambda) \) is a L-fuzzy \( \omega \)-clopen set of \((X, T)\), for every L-fuzzy \( \omega \)-open set \( \lambda \) of \((Y, S)\).
Proposition 5.2.6

Let \((X, T)\) be an L-fuzzy topological space. Then the following conditions are equivalent:

(a) \((X, T)\) is an L-fuzzy \(\omega\)-extremally disconnected space.

(b) If \(g, h : X \to \mathbb{R}(L)\) where \(g\) is lower L-fuzzy \(\omega\)-continuous, \(h\) is upper L-fuzzy \(\omega\)-continuous, then there exists \(f \in \text{CL}_\omega (X)\) such that 
\[g \leq f \leq h.\] \([\text{CL}_\omega (X) = \text{collection of all strong L-fuzzy \(\omega\)-continuous mappings on } X \text{ with values in } \mathbb{R}(L)].\]

(c) If \(\lambda\) is L-fuzzy \(\omega\)-closed and \(\mu\) is L-fuzzy \(\omega\)-open sets such that \(\mu \leq \lambda\), then there exists a strong L-fuzzy \(\omega\)-continuous mapping \(f : X \to I(L)\) such that 
\[\mu \leq (1 - L_1 f) \leq R_\alpha f \leq \lambda.\]

Proof

(a) \(\Rightarrow\) (b) Define \(H_r = L_r h\) and \(G_r = (1 - R_r) g\), \(r \in Q\). Thus, we have two monotone increasing families of respectively L-fuzzy \(\omega\)-open sets and L-fuzzy \(\omega\)-closed sets of \((X, T)\). Moreover \(H_r \leq G_s\) if \(r < S\). By Proposition 5.2.3, there exists a monotone increasing family \(\{F_r\}_{r \in Q}\) \((X, T)\) such that \(L\omega\text{-cl} (H_r) \leq F_S\) and \(F_r \leq L\omega\text{-int} (G_S)\) whenever \(r < S\). Letting \(V_t = \bigwedge_{r \in t} (1 - F_r)\) for all \(t \in R\), we define a monotone decreasing family \(\{V_t / t \in R\} \subset L^X\). Moreover we have \(L\omega\text{-cl} (V_t) \leq L\omega\text{-int} (V_s)\) whenever \(s < t\). We have
\[
\bigvee_{t \in R} V_t \geq \bigwedge_{r \in t} (1 - G_r)
\]

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\[
\begin{align*}
&= \bigvee_{t \in R} \bigwedge_{r < t} g^{-1}(R_r) \\
&= g^{-1}(\bigvee_{t \in R} R_t) = 1.
\end{align*}
\]

Similarly, \( \bigwedge_{t \in R} V_t = 0 \). We now define a mapping \( f : X \to R(L) \) satisfying the required properties. Let \( f (x) (t) = V_t (x) \), for all \( x \in X \) and \( t \in R \). By the above discussion it follows that \( f \) is well defined. To prove \( f \) is strong \( L \)-fuzzy \( \omega \)-continuous, we observe that

\[
\bigvee_{s > t} V_s = \bigvee_{s > t} L\omega\text{-int} (V_s) \quad \text{and} \quad \bigwedge_{s < t} V_s = \bigwedge_{s < t} L\omega\text{-cl} (V_s).
\]

Then,

\[
f^{-1}(R_t) = \bigvee_{s > t} V_s
\]

is \( L\omega \)-fuzzy \( \omega \)-clopen.

And

\[
f^{-1}(L_t') = \bigwedge_{s < t} V_s
\]

is \( L\omega \)-fuzzy \( \omega \)-clopen.

Therefore, \( f \) is strong \( L \)-fuzzy \( \omega \)-continuous. To conclude the proof it remains to show that \( g \leq f \leq h \). That is, \( g^{-1}(1 - L_t) \leq f^{-1}(1 - L_t) \leq h^{-1}(1 - L_t) \) and \( g^{-1}(R_t) \leq f^{-1}(R_t) \leq h^{-1}(R_t) \), for each \( t \in R \). We have

\[
g^{-1}(1 - L_t) = \bigwedge_{s < t} g^{-1}(1 - L_s)
\]

\[
= \bigwedge_{s < t} \bigwedge_{r < a} g^{-1}(R_r)
\]

\[
= \bigwedge_{s < t} \bigwedge_{r < a} (1 - G_r)
\]

\[
\leq \bigwedge_{s < t} \bigwedge_{r < a} (1 - F_r)
\]

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\[ V_s = f^{-1}(1 - L_t) \]

and

\[ f^{-1}(1 - L_t) = \bigwedge_{s:t} V_s \]

\[ = \bigwedge_{s:t} (1 - F_r) \]

\[ \leq \bigwedge_{s:t} (1 - H_t) \]

\[ = \bigwedge_{s:t} h^{-1}(1 - L_t) \]

\[ = h^{-1}(1 - L_t) \]

Similarly,

\[ g^{-1}(R_t) = \bigvee_{s:t} g^{-1}(R_s) \]

\[ = \bigvee_{s:t} \bigvee_{r:s} g^{-1}(R_r) \]

\[ = \bigvee_{s:t} (1 - G_t) \]

\[ \leq \bigvee_{s:t} (1 - F_r) \]

\[ = \bigvee_{s:t} V_s \]

\[ = f^{-1}(R_t) \]

and

\[ f^{-1}(R_t) = \bigvee_{s:t} V_s \]

\[ = \bigvee_{s:t} \bigwedge_{r:s} (1 - F_r) \]

\[ \leq \bigvee_{s:t} (1 - H_t) \]

\[ = \bigvee_{s:t} h^{-1}(1 - L_t) \]

\[ = \bigvee_{s:t} h^{-1}(R_s) = h^{-1}(R_t). \]

Thus, (b) is proved.
(b) ⇒ (c) Suppose λ is L-fuzzy ω-closed and μ is L-fuzzy ω-open such that μ ≤ λ. Then, \( \chi_\mu \leq \chi_\lambda \) where \( \chi_\mu, \chi_\lambda \) are lower and upper L-fuzzy ω-continuous respectively. Hence by (b), there exists a strong L-fuzzy ω-continuous mapping \( f : X \to R(L) \) such that, \( \chi_\mu \leq f \leq \chi_\lambda \). Clearly, \( f(x) \in I(L) \), for all \( x \in X \) and \( \mu = (1 - L_1) \chi_\mu \leq (1 - L_1) f \leq R_0 f \leq R_0 \chi_\lambda = \lambda \). Therefore, \( \mu \leq (1 - L_1) f \leq R_0 f \leq \lambda \).

(c) ⇒ (a) \( (1 - L_1) f \) and \( R_0 f \) are L-fuzzy ω-clopen sets. By Proposition 5.2.2, \( (X, T) \) is an L-fuzzy ω-extremally disconnected space.

5.3 TIETZE EXTENSION THEOREM

In this section, Tietze extension theorem for L-fuzzy ω-extremally disconnected spaces is discussed.

Proposition 5.3.1

Let \( (X, T) \) be an L-fuzzy ω-extremally disconnected space and let \( A \subset X \) be such that \( \chi_A \) is L-fuzzy ω-open. Let \( f : (A, T/A) \to I(L) \) be strong L-fuzzy ω-continuous. Then \( f \) has a strong L-fuzzy ω-continuous extension over \( (X, T) \).

Proof

Let \( g, h : X \to I(L) \) be such that \( g = f = h \) on \( A \) and \( g(x) = 0 \), \( h(x) = 1 \) if \( x \notin A \). We now have

\[
R_t g = \begin{cases}
\mu_t \wedge \chi_A & \text{if } t \geq 0 \\
1 & \text{if } t < 0
\end{cases}
\]

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where $\mu_t$ is L-fuzzy $\omega$-open and is such that $\mu_t/A = R_t f$ and

$$L_t h = \begin{cases} 
\lambda_t \land \chi_A & \text{if } t \leq 1 \\
1 & \text{if } t > 1
\end{cases}$$

where $\lambda_t$ is L-fuzzy $\omega$-open / L-fuzzy $\omega$-closed and is such that $\lambda_t/A = L_t f$.

Thus, $g$ is lower L-fuzzy $\omega$-continuous, $h$ is upper / L-fuzzy $\omega$-continuous and $g \leq h$. By Proposition 5.2.6, there is a strong L-fuzzy $\omega$-continuous mapping $F : X \to I(L)$ such that $g \leq F \leq h$. Hence, $F \equiv f$ on $A$. 