CHAPTER – 8

NOTIONS VIA GENERALIZED FUZZY PRE-OPEN SETS

Fuzzy pre-open sets and fuzzy pre-closed sets were introduced by Singal and Prakash [67]. Generalization of fuzzy pre-continuous functions and generalized fuzzy-pre-irresolute functions are introduced in [14] and [80] respectively. In this chapter, the concepts of gfpre-border, gfpre-Exterior and gfpre-Frontier are introduced. Some interesting properties and interrelations among the concepts introduced are established with relevant examples. Further, some characterizations concerning generalized fuzzy pre-continuous functions and fuzzy gc-pre-irresolute functions are studied.
8.1 PROPERTIES OF GFPRE-OPEN SETS

In this section, the concepts of generalized fuzzy pre-border, generalized fuzzy pre-exterior and generalized fuzzy pre-frontier are introduced. Some interesting properties and characterizations of the concepts introduced are investigated.

**Definition 8.1.1**

Generalized fuzzy pre-closure of \( \lambda \) (briefly, gfpcl(\( \lambda \))) is defined as
\[
\text{gfpcl}(\lambda) = \wedge \{ \mu / \lambda \leq \mu, \mu \text{ is gfpre-closed} \}.
\]

**Definition 8.1.2**

Generalized fuzzy pre-interior of \( \lambda \) (briefly, gf pint(\( \lambda \))) is defined as
\[
\text{gf pint}(\lambda) = \vee \{ \mu / \mu \leq \lambda, \mu \text{ is gfpre-open} \}.
\]

**Definition 8.1.3**

Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. A mapping \(f : (X, T) \to (Y, S)\) is called gfpre-open (resp. gfpre-closed) if the image of every gfpre-open (resp. gfpre-closed) set in \((X, T)\) is gfpre-open (resp. gfpre-closed) in \((Y, S)\).

**Proposition 8.1.1**

Let \((X, T)\) be a fuzzy topological space. For any two fuzzy sets \(\lambda, \mu \in \Gamma^X\), the following statements hold:

(a) \( \text{gf pint}(\lambda) \) is the largest gfpre-open set such that \( \text{gf pint}(\lambda) \leq \lambda \).

(b) If \( \lambda \) is gfpre-open, then \( \lambda = \text{gf pint}(\lambda) \).

(c) If \( \lambda \) is gfpre-open, then \( \text{gf pint}(\text{gf pint}(\lambda)) = \text{gf pint}(\lambda) \).

(d) \( 1 - \text{gf pint}(\lambda) = \text{gfpcl}(1 - \lambda) \).

(e) \( 1 - \text{gf pcl}(\lambda) = \text{gf pint}(1 - \lambda) \).
(f) If \( \lambda \leq \mu \), then \( \text{gfpint}(\lambda) \leq \text{gfpint}(\mu) \).

(g) \( \text{gfpint}(\lambda \wedge \mu) = \text{gfpint}(\lambda \wedge \mu) \).

(h) \( \text{gfpint}(\lambda \lor \mu) = \text{gfpint}(\lambda \lor \mu) \).

(i) If \( \lambda \leq \mu \), then \( \text{gfpcl}(\lambda) \leq \text{gfpcl}(\mu) \).

**Proof**

(a) and (b) follows from the definitions and (c) follows from (b).

(d) \( \text{gfpcl}(1 - \lambda) = \lambda \{ \mu / \mu \geq 1 - \lambda \text{ and } \mu \text{ is gfp-re-closed} \} \)

= \( 1 - \lor \{ 1 - \mu / 1 - \mu \leq \lambda \text{ and } (1 - \mu) \text{ is gfp-re-open} \} \). Therefore, \( \text{gfpcl}(1 - \lambda) = 1 - \text{gfpint}(\lambda) \).

(e) \( \text{gfpint}(1 - \lambda) = \lor \{ \mu / \mu \leq 1 - \lambda \text{ and } \mu \text{ is gfp-re-open} \} = \)

\( 1 - \land \{ 1 - \mu / 1 - \mu \geq \lambda \text{ and } (1 - \mu) \text{ is gfp-re-closed} \} = 1 - \text{gfpcl}(\lambda) \). Therefore, \( 1 - \text{gfpcl}(\lambda) = \text{gfpint}(1 - \lambda) \).

(f) Let \( \lambda \leq \mu \). Then,

\( \text{gfpint}(\lambda) = \lor \{ \delta / \delta \leq \lambda \text{ and } \delta \text{ is gfp-re-open} \} \)

\( \leq \lor \{ \delta / \delta \leq \mu \text{ and } \delta \text{ is gfp-re-open} \} \)

= \( \text{gfpint}(\mu) \).

Therefore, \( \text{gfpint}(\lambda) \leq \text{gfpint}(\mu) \).

(g) \( \text{gfpint}(\lambda \wedge \mu) = \lor \{ \delta / \delta \leq \lambda \wedge \mu \text{ and } \delta \text{ is gfp-re-open} \} \)

= \( (\lor \{ \delta / \delta \leq \lambda \text{ and } \delta \text{ is gfp-re-open} \}) \land (\lor \{ \delta / \delta \leq \mu \text{ and } \delta \text{ is gfp-re-open} \}) \)

= \( (\text{gfpint}(\lambda)) \land (\text{gfpint}(\mu)) \).

Hence, \( \text{gfpint}(\lambda \wedge \mu) = (\text{gfpint}(\lambda)) \land (\text{gfpint}(\mu)) \).

(h) \( \text{gfpint}(\lambda \lor \mu) = \lor \{ \delta / \delta \leq \lambda \lor \mu \text{ and } \delta \text{ is gfp-re-open} \} \)
\[
(\vee \{ \delta / \delta \leq \lambda \text{ and } \delta \text{ is gfpre-open} \}) \wedge (\vee \{ \delta / \delta \leq \mu \text{ and } \delta \text{ is gfpre-open} \}).
\]

Therefore, \( \text{gf pint (} \lambda \vee \mu \text{)} = (\text{gf pint (} \lambda \text{)}) \vee (\text{gf pint (} \mu \text{)}). \)

(i) Let \( \lambda \leq \mu \). Then,

\[
\text{gfcl (} \lambda \text{)} = \bigwedge \{ \delta / \delta \geq \lambda \text{ and } \delta \text{ is gfpre-closed} \}
\]

\[
\leq \bigwedge \{ \delta / \delta \geq \mu \text{ and } \delta \text{ is gfpre-closed} \}
\]

\[
= \text{gfcl (} \mu \text{)}. \]

Therefore, \( \text{gfcl (} \lambda \text{)} \leq \text{gfcl (} \mu \text{)}. \)

**Definition 8.1.4**

Let \((X, T)\) be a fuzzy topological space. For any fuzzy set \(\lambda \in I^X\), generalised fuzzy border of \(\lambda\) denoted by \(\text{gfb (} \lambda \text{)}\) is defined as

\[
\text{gfb (} \lambda \text{)} = \lambda - \text{gf pint (} \lambda \text{)}. \]

**Definition 8.1.5**

Let \((X, T)\) be a fuzzy topological space. For any fuzzy set \(\lambda \in I^X\), fuzzy pre-border of \(\lambda\) denoted by \(\text{fpb (} \lambda \text{)}\) is defined as

\[
\text{fpb (} \lambda \text{)} = \lambda - \text{fp pint (} \lambda \text{)}. \]

**Definition 8.1.6**

Let \((X, T)\) be a fuzzy topological space. For any fuzzy set \(\lambda \in I^X\), generalized fuzzy pre-border of \(\lambda\) denoted by \(\text{gf pb (} \lambda \text{)}\) is defined as

\[
\text{gf pb (} \lambda \text{)} = \lambda - \text{gf pint (} \lambda \text{)}. \]

**Proposition 8.1.2**

Let \((X, T)\) be a fuzzy topological space. For any fuzzy set \(\lambda \in I^X\), the following statements hold:
(a) \( \text{gfp b} (\lambda) \leq \text{fpb} (\lambda) \).

(b) If \( \lambda \) is gfpre-open, then \( \text{gfpb} (\lambda) = 0 \).

(c) \( \text{gfpb} (\lambda) \leq \text{gfpcl} (1 - \lambda) \).

(d) \( \text{gfpint} (\text{gfpb} (\lambda)) \leq \lambda \).

(e) \( \text{gfpb} (\lambda \lor \mu) \leq (\text{gfpb} (\lambda)) \lor (\text{gfpb} (\mu)) \).

(f) \( \text{gfpb} (\lambda \land \mu) \geq (\text{gfpb} (\lambda)) \land (\text{gfpb} (\mu)) \).

**Proof**

(a) Let \( \lambda \) be any fuzzy set in \((X, T)\).

Since \( \text{fpint} (\lambda) \leq \text{gfpint} (\lambda) \),

\[ \lambda - \text{gfpint} (\lambda) \leq \lambda - \text{fpint} (\lambda) . \]

Therefore,

\[ \text{gfpb} (\lambda) \leq \text{fpb} (\lambda) . \]

(b) Let \( \lambda \) be a gfpre-open set. Then,

\[ \lambda = \text{gfpint} (\lambda) . \]

Thus,

\[ \text{gfpb} (\lambda) = 0 . \]

(c) \( \text{gfpb} (\lambda) = \lambda - \text{gfpint} (\lambda) \)

\[ = \lambda - (1 - \text{gfpcl} (1 - \lambda)) \]

\[ \leq 1 - 1 + \text{gf pcl} (1 - \lambda) \]

\[ = \text{gf pcl} (1 - \lambda) . \]

Hence, \( \text{gfpb} (\lambda) \leq \text{gf pcl} (1 - \lambda) . \)

(d) \( \text{gfpint} (\text{gfpb} (\lambda)) = \text{gfpint} (\lambda - \text{gfpint} (\lambda)) \)

\[ \leq \lambda - \text{gfpint} (\lambda) \]

\[ \leq \lambda , \text{ by (a) of Proposition 8.1.1} . \]
Therefore,
\[
gfp ( \lambda ) \leq \lambda.
\]
\textbf{(e) } \quad \text{gfp (} \lambda \text{ } \lor \text{ } \mu \text{)} = ( \lambda \lor \mu ) - \text{gfp (} \lambda \lor \mu \text{)}
\[
= ( \lambda \lor \mu ) - ( \text{gfp (} \lambda \text{)} \lor ( \text{gfp (} \mu \text{)})
\]
\[
\leq ( \lambda - \text{gfp (} \lambda \text{)} ) \lor ( \mu - \text{gfp (} \mu \text{)})
\]
\[
= ( \text{gfp (} \lambda \text{)} \lor ( \text{gfp (} \mu \text{)})).
\]
Therefore,
\[
gfp ( \lambda \lor \mu ) \leq ( \text{gfp (} \lambda \text{)} \lor ( \text{gfp (} \mu \text{)})).
\]
\textbf{(f) } \quad \text{gfp (} \lambda \land \mu \text{)} = ( \lambda \land \mu ) - \text{gfp (} \lambda \land \mu \text{)}
\[
= ( \lambda \land \mu ) - ( \text{gfp (} \lambda \text{)} \land ( \text{gfp (} \mu \text{)})
\]
\[
\geq ( \lambda - \text{gfp (} \lambda \text{)} ) \land ( \mu - \text{gfp (} \mu \text{)})).
\]
Therefore,
\[
gfp ( \lambda \land \mu ) \geq ( \text{gfp (} \lambda \text{)} \land ( \text{gfp (} \mu \text{)})).
\]

\textbf{Definition 8.1.7}

Let ( X, T ) be a fuzzy topological space. For any fuzzy set \( \lambda \in I^X \),
generalized fuzzy Frontier of \( \lambda \) denoted by \( \text{gfFr (} \lambda \text{)} \) is defined as
\[
\text{gfFr (} \lambda \text{)} = \text{gfcl (} \lambda \text{)} - \text{gfint (} \lambda \text{)}.
\]

\textbf{Definition 8.1.8}

Let ( X, T ) be a fuzzy topological space. For any fuzzy set \( \lambda \in I^X \),
fuzzy pre-Frontier of \( \lambda \) denoted by \( \text{fpFr (} \lambda \text{)} \) is defined as
\[
\text{fpFr (} \lambda \text{)} = \text{fpcl (} \lambda \text{)} - \text{fpint (} \lambda \text{)}.
\]
**Definition 8.1.9**

Let \(( X, T )\) be a fuzzy topological space. For any fuzzy set \( \lambda \in \hat{I}^X \), generalized fuzzy pre-Frontier of \( \lambda \) denoted by \( \text{gfpFr} ( \lambda ) \) is defined as \( \text{gfpFr} ( \lambda ) = \text{gfpcl} ( \lambda ) - \text{gfpint} ( \lambda ) \).

**Proposition 8.1.3**

Let \(( X, T )\) be a fuzzy topological space. For any fuzzy set \( \lambda \in \hat{I}^X \), the following statements hold:

(a) \( \text{gfpFr} ( \lambda ) \leq \text{fpFr} ( \lambda ) \).

(b) \( \text{gfpb} ( \lambda ) \leq \text{gfpFr} ( \lambda ) \).

(c) \( \text{gfpFr} ( 1 - \lambda ) = \text{gfpFr} ( \lambda ) \).

(d) \( \text{gfpFr} ( \text{gfpint} ( \lambda ) ) \leq \text{gfpFr} ( \lambda ) \).

(e) \( \text{gfpFr} ( \text{gfpcl} ( \lambda ) ) \leq \text{gfpFr} ( \lambda ) \).

(f) \( \lambda - \text{gfpFr} ( \lambda ) \leq \text{gfpint} ( \lambda ) \).

(g) \( \text{gfpFr} ( \lambda \lor \mu ) \leq ( \text{gfpFr} ( \lambda ) ) \lor ( \text{gfpFr} ( \mu ) ) \).

(h) \( \text{gfpFr} ( \lambda \land \mu ) \geq ( \text{gfpFr} ( \lambda ) ) \land ( \text{gfpFr} ( \mu ) ) \).

**Proof**

(a) \( \text{gfpFr} ( \lambda ) = \text{gfpcl} ( \lambda ) - \text{gfpint} ( \lambda ) \)

\[ \leq \text{fpcl} ( \lambda ) - \text{fpint} ( \lambda ) \]

\[ = \text{fpFr} ( \lambda ). \]

Therefore,

\( \text{gfpFr} ( \lambda ) \leq \text{fpFr} ( \lambda ). \)

(b) \( \text{gfpb} ( \lambda ) = \lambda - \text{gfpint} ( \lambda ) \)

\[ \leq \text{gfpcl} ( \lambda ) - \text{gfpint} ( \lambda ), \text{ since } \lambda \leq \text{gfpcl} ( \lambda ) \]
= gfpFr (\( \lambda \)).

Therefore,

\[
gfpb (\lambda) \leq gfpFr (\lambda).
\]

\(\textbf{(c) } gfpFr (\lambda) = gfpcl (\lambda) - gfpint (\lambda)\)

\[
= gfpcl (\lambda) - (1 - gfpcl (1 - \lambda))
\]

\[
= gfpcl (\lambda) - 1 + gfpcl (1 - \lambda)
\]

\[
= -gfpint (1 - \lambda) + gfpcl (1 - \lambda)
\]

\[
= gfpFr (1 - \lambda).
\]

Therefore,

\[
gfpFr (1 - \lambda) = gfpFr (\lambda).
\]

\(\textbf{(d) } gfpFr (gfpint (\lambda)) = gfpcl (gfpint (\lambda)) - gfpint (gfpint (\lambda))\)

\[
\leq gfpcl (\lambda) - gfpint (\lambda)
\]

\[
= gfpFr (\lambda).
\]

Therefore,

\[
gfpFr (gfpint (\lambda)) \leq gfpFr (\lambda).
\]

\(\textbf{(e) } gfpFr (gfpcl (\lambda)) = gfpcl (gfpcl (\lambda)) - gfpint (gfpcl (\lambda))\)

\[
= gfpcl (\lambda) - gfpint (gfpcl (\lambda))
\]

\[
\geq gfpcl (\lambda) - gfpint (\lambda)
\]

\[
= gfpFr (\lambda).
\]

Therefore,

\[
gfpFr (gfpcl (\lambda)) \leq gfpFr (\lambda).
\]

\(\textbf{(f) } \lambda - gfpFr (\lambda) = \lambda - (gfpcl (\lambda) - gfpint (\lambda))\)

\[
\leq gfpcl (\lambda) - gfpcl (\lambda) + gfpint (\lambda)
\]

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\[
\begin{align*}
\lambda - \text{gfpFr} (\lambda) &\leq \text{gfpint} (\lambda).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\lambda - \text{gfpFr} (\lambda) &\leq \text{gfpint} (\lambda).
\end{align*}
\]

\textbf{(g)} \quad \text{gfpFr} (\lambda \lor \mu) = \text{gfpcl} (\lambda \lor \mu) - \text{gfpint} (\lambda \lor \mu)
\]

\[
\begin{align*}
&= \text{gfpcl} (\lambda \lor \mu) - (\text{gfpint} (\lambda) \lor \text{gfpint} (\mu)) \\
&= ((\text{gfpcl} (\lambda)) \lor (\text{gfpcl} (\mu))) \\
&\quad - ((\text{gfpint} (\lambda)) \lor (\text{gfpint} (\mu))) \\
&\leq (\text{gfpcl} (\lambda) - \text{gfpint} (\lambda)) \\
&\quad \lor (\text{gfpcl} (\mu) - \text{gfpint} (\mu)) \\
&= (\text{gfpFr} (\lambda) \lor \text{gfpFr} (\mu)).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\text{gfpFr} (\lambda \lor \mu) &\leq (\text{gfpFr} (\lambda) \lor \text{gfpFr} (\mu)).
\end{align*}
\]

\textbf{(h)} \quad \text{gfpFr} (\lambda \land \mu) = \text{gfpcl} (\lambda \land \mu) - \text{gfpint} (\lambda \land \mu)
\]

\[
\begin{align*}
&= \text{gfpcl} (\lambda \land \mu) - (\text{gfpint} (\lambda) \land \text{gfpint} (\mu)) \\
&= ((\text{gfpcl} (\lambda)) \land (\text{gfpcl} (\mu))) \\
&\quad - ((\text{gfpint} (\lambda)) \land (\text{gfpint} (\mu))) \\
&\geq (\text{gfpcl} (\lambda) - \text{gfpint} (\lambda)) \\
&\quad \land (\text{gfpcl} (\mu) - \text{gfpint} (\mu)) \\
&= (\text{gfpFr} (\lambda) \land \text{gfpFr} (\mu)).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\text{gfpFr} (\lambda \land \mu) &\geq (\text{gfpFr} (\lambda) \land \text{gfpFr} (\mu)).
\end{align*}
\]

\textbf{Definition 8.1.10}

Let \((X, T)\) be a fuzzy topological space and let \(\lambda \in I^X\). Then
generalised fuzzy Exterior of \( \lambda \) denoted by \( \text{gfExt} ( \lambda ) \) is defined as \( \text{gfExt} ( \lambda ) = \text{gfint} (1 - \lambda) \).

**Definition 8.1.11**

Let \((X, \mathcal{T})\) be a fuzzy topological space and let \( \lambda \in \mathcal{I}^X \). Then fuzzy pre-Exterior of \( \lambda \) denoted by \( \text{fpExt} ( \lambda ) \) is defined as \( \text{fpExt} ( \lambda ) = \text{fpint} (1 - \lambda) \).

**Definition 8.1.12**

Let \((X, \mathcal{T})\) be a fuzzy topological space and let \( \lambda \in \mathcal{I}^X \). Then generalized fuzzy pre-Exterior of \( \lambda \) denoted by \( \text{gfpExt} ( \lambda ) \) is defined as \( \text{gfpExt} ( \lambda ) = \text{gfpint} (1 - \lambda) \).

**Proposition 8.1.4**

Let \((X, \mathcal{T})\) be a fuzzy topological space. For a fuzzy set \( \lambda \in \mathcal{I}^X \), the following statements hold:

- (a) \( \text{fpExt} ( \lambda ) \leq \text{gfpExt} ( \lambda ) \).
- (b) \( \text{gfpExt} ( \lambda ) = 1 - \text{gfpcl} ( \lambda ) \).
- (c) \( \text{gfpExt} ( \text{gfpExt} ( \lambda ) ) = \text{gfpint} ( \text{gfpcl} ( \lambda ) ) \).
- (d) If \( \lambda \leq \mu \), then \( \text{gfpExt} ( \lambda ) \geq \text{gfpExt} ( \mu ) \).
- (e) \( \text{gfpExt} (1) = 0 \).
- (f) \( \text{gfpExt} (0) = 1 \).
- (g) \( \text{gfpint} ( \lambda ) \leq \text{gfpExt} ( \text{gfpExt} ( \lambda ) ) \).
- (h) \( \text{gfpExt} ( \lambda \lor \mu ) = ( \text{gfpExt} ( \lambda ) ) \land ( \text{gfpExt} ( \mu ) ) \).
- (i) \( \text{gfpExt} ( \lambda \land \mu ) = ( \text{gfpExt} ( \lambda ) ) \lor ( \text{gfpExt} ( \mu ) ) \).
Proof

(a) Since \( \text{gfpcl} (\lambda) \leq \text{fpcl} (\lambda) \),
\[ 1 - \text{gfpcl} (\lambda) \geq 1 - \text{fpcl} (\lambda) \]
which implies,
\[ \text{gfpint} (1 - \lambda) \geq \text{fpint} (1 - \lambda). \]
Therefore by definition,
\[ \text{gfp Ext} (\lambda) \geq \text{fp Ext} (\lambda). \]

(b) The proof follows from the definitions.

(c) By definition,
\[ \text{gfp Ext} (\text{gfp Ext}(\lambda)) = \text{gfpint} (1 - \text{gfp Ext}(\lambda)) \]
\[ = \text{gfpint} (1 - \text{gfp int}(1 - \lambda)). \]
Therefore,
\[ \text{gfp Ext} (\text{gfp Ext}(\lambda)) = \text{gfpint} (\text{gfpcl}(\lambda)). \]

(d) Let \( \lambda \leq \mu \). Then by (i) of Proposition 8.1.1,
\[ \text{gfpcl} (\lambda) \leq \text{gfpcl} (\mu). \]
Therefore,
\[ 1 - \text{gfpcl} (\lambda) \geq 1 - \text{gfpcl} (\mu). \]
By (e) of Proposition 8.1.1,
\[ \text{gfpint} (1 - \lambda) \geq \text{gfpint} (1 - \mu). \]
Hence,
\[ \text{gfpExt} (\lambda) \geq \text{gfpExt} (\mu). \]

(e) By (b), \( \text{gfpExt}(1) = 1 - \text{gfpcl}(1) \)
\[ = 1 - 1 \]
\[ = 0. \]
Therefore,

$$\text{gfpExt}(1) = 0.$$  

**(f)** By (b), \(\text{gfpExt}(0) = 1 - \text{gfpcl}(0)\)

\[= 1 - 0\]

\[= 1.\]

Therefore,

$$\text{gfpExt}(0) = 1.$$  

**(g)** Since \(\lambda \leq \text{gfpcl}(\lambda)\), by (f) of Proposition 8.1.1,

$$\text{gfpint}(\lambda) \leq \text{gfpint}(\text{gfpcl}(\lambda)).$$

Then by (c),

$$\text{gfpint}(\lambda) \leq \text{gfpExt}(\text{gfpExt}(\lambda)).$$

**(h)** \(\text{gfpExt}(\lambda \lor \mu) = \text{gfpint}(1 - \lambda \lor \mu)\)

\[= \text{gfpint}((1 - \lambda) \land (1 - \mu))\]

\[= (\text{gfpint}(1 - \lambda)) \land (\text{gfpint}(1 - \mu))\]

\[= (\text{gfpExt}(\lambda)) \land (\text{gfpExt}(\mu)).\]

Therefore,

$$\text{gfpExt}(\lambda \lor \mu) = (\text{gfpExt}(\lambda)) \land (\text{gfpExt}(\mu)).$$

**(i)** \(\text{gfpExt}(\lambda \land \mu) = \text{gfpint}(1 - \lambda \land \mu)\)

\[= \text{gfpint}((1 - \lambda) \lor (1 - \mu))\]

\[= (\text{gfpint}(1 - \lambda)) \lor (\text{gfpint}(1 - \mu))\]

\[= (\text{gfpExt}(\lambda)) \lor (\text{gfpExt}(\mu)).\]

Therefore,

$$\text{gfpExt}(\lambda \land \mu) = (\text{gfpExt}(\lambda)) \lor (\text{gfpExt}(\mu)).$$
8.2 CHARACTERIZATIONS OF FUZZY GC-PRE-IRRESOLUTE AND GFPRE-CONTINUOUS MAPPINGS

In this section, some characterizations of gc-pre-irresolute and gf pre-continuous mappings are studied.

**Proposition 8.2.1**

Let (X, T) and (Y, S) be any two fuzzy topological spaces. Let f : (X, T) → (Y, S) be any mapping. Then the following conditions are equivalent:

(a) f is fuzzy gc-pre-irresolute .

(b) For every fuzzy set λ in (X, T), f (gfpcl (λ)) ⊆ gfpcl (f (λ)).

(c) For every fuzzy set μ in (Y, S), gfpcl (f⁻¹(μ)) ⊆ f⁻¹(gfpcl (μ)).

**Proof**

\((a) \Rightarrow (b)\) Let λ be a fuzzy set in (X, T). Since gfpcl (f (λ)) is a gfpre-closed set in (Y, S), by (a), f⁻¹(gfpcl (f (λ))) is gfpre-closed in (X, T). Therefore, gfpcl (f⁻¹(gfpcl (f (λ))))

\[= f⁻¹(gfpcl(f(λ))). \tag{8.2.1} \]

Since \(λ \leq f⁻¹(f(λ))\), gfpcl (λ) ⊆ gfpcl (f⁻¹(f(λ))).

Since \(f(λ) \leq gfpcl(f(λ))\) and by (8.2.1),

\[gfpcl(λ) \leq gfpcl(f⁻¹(gfpcl(f(λ))))\]

\[= f⁻¹(gfpcl(f(λ))).\]

Hence, \(f(gfpcl(λ)) \leq gfpcl(f(λ)).\)
(b) ⇒ (c) Let $\mu$ be a fuzzy set in $(Y, S)$. As $f^{-1}(\mu)$ is a fuzzy set in $(X, T)$, by (b),

$$f(\text{gfpc} \ f^{-1}(\mu)) \leq \text{gfpc} \ f(f^{-1}(\mu))$$

$$\leq \text{gfpc} \ \mu.$$ 

That is,

$$f(\text{gfpc} \ f^{-1}(\mu)) \leq \text{gfpc} \ \mu.$$ 

Therefore, $f^{-1}(f(\text{gfpc} \ f^{-1}(\mu))) \leq f^{-1}(\text{gfpc} \ \mu)$. 

Hence,

$$\text{gfpc} \ f^{-1}(\mu) \leq f^{-1}(\text{gfpc} \ \mu).$$

(c) ⇒ (a) Let $\gamma$ be a gfpre-closed set in $(Y, S)$. Then,

$$\text{gfpc} \ \gamma = \gamma.$$ 

By (c), $\text{gfpc} \ f^{-1}(\gamma) \leq f^{-1}(\text{gfpc} \ \gamma)$

$$f^{-1}(\gamma) = \text{gfpc} \ f^{-1}(\gamma).$$

But $f^{-1}(\gamma) \leq \text{gfpc} \ f^{-1}(\gamma)$.  

Therefore by (8.2.2) and (8.2.3),

$$f^{-1}(\gamma) = \text{gfpc} \ f^{-1}(\gamma)$$

which implies that $f^{-1}(\gamma)$ is a gfpre-closed set in $(X, T)$. Hence, $f$ is fuzzy gc-pre-irresolute.

**Proposition 8.2.2**

Let $(X, T)$ and $(Y, S)$ be any two fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ be any map. Then $f$ is gfpre-closed iff

$$\text{gfpc} \ f(\lambda) \leq f(\text{gfpc} \ \lambda),$$

for each fuzzy set $\lambda$ in $(X, T)$.

**Proof**

Let $\lambda$ be any fuzzy set in $(X, T)$. Suppose that $f$ is a gfpre-closed map. Then, since $\text{gfpc} \ \lambda$ is a gfpre-closed set in $(X, T)$,

$$f(\text{gfpc} \ \lambda)$$

is gfpre-closed in $(Y, S)$. Therefore,
\[
gfpcl(f(gfpcl(\lambda))) = f(gfpcl(\lambda)).
\]

Since \(\lambda \leq \gfpcl(\lambda)\), it follows that
\[
f(\lambda) \leq f(gfpcl(\lambda)).
\]

Therefore, \(\gfpcl(f(\lambda)) \leq \gfpcl(f(gfpcl(\lambda)))\)
\[
= f(gfpcl(\lambda)).
\]

Hence, \(\gfpcl(f(\lambda)) \leq f(gfpcl(\lambda))\), for each fuzzy set \(\lambda\) in \((X, T)\).

Conversely, suppose that \(\gfpcl(f(\lambda)) \leq f(gfpcl(\lambda))\), for each fuzzy set \(\lambda\) in \((X, T)\).

Let \(\lambda\) be a gfpcl-closed set. Then,
\[
gfpcl(\lambda) = \lambda.
\]

Therefore, \(f(gfpcl(\lambda)) = f(\lambda)\)
\[
\leq gfpcl(f(\lambda)). \tag{8.2.5}
\]

Hence from (8.2.4) and (8.2.5),
\[
f(\lambda) = f(gfpcl(\lambda)) = gfpcl(f(\lambda)),\text{ which implies that } f(\lambda) \text{ is a gfpcl-closed set in } (Y, S). \text{ Thus, } f \text{ is a gfpcl-closed map.}
\]

**Proposition 8.2.3**

Let \((X, T)\) be any fuzzy topological space and let \((Y, S)\) be a fuzzy pre-\(T_{1/2}\) space. Let \(f: (X, T) \rightarrow (Y, S)\) be a bijective map. Then the following conditions are equivalent:

(a) \(f\) and \(f^{-1}\) are fuzzy gc-pre-irresolutes.

(b) \(f\) is gfpcl-continuous and gfpcl-open.

(c) \(f\) is gfpcl-continuous and gfpcl-closed.

(d) \(f(gfpcl(\lambda)) = gfpcl(f(\lambda))\), for each fuzzy set \(\lambda\) in \((X, T)\).
Proof

(a) ⇒ (b) Let $\mu$ be a gfpre-open set in $(X, T)$. Since $f^{-1}$ is gc-pre-irresolute, $(f^{-1})^{-1}(\mu)$ is a gfpre-open set in $(Y, S)$, proving that $f$ is gfpre-open. Let $\sigma$ be a fuzzy pre-open set in $(Y, S)$. Then, it is gfpre-open and hence by Proposition 10 in [79], $f^{-1}(\sigma)$ is gfpre-open in $(X, T)$, proving that $f$ is gfpre-continuous, by Proposition 9 in [79].

(b) ⇒ (c) To prove $f$ is gfpre-closed. Let $\mu$ be a gfpre-closed set in $(X, T)$. Then, $1 - \mu$ is gfpre-open in $(X, T)$. By (b), $f(1 - \mu)$ is gfpre-open in $(Y, S)$. Therefore,

$$1 - f(\mu) = f(1 - \mu)$$

implies that $f(\mu)$ is gfpre-closed in $(Y, S)$. Hence, $f$ is gfpre-closed.

(c) ⇒ (d) Let $\lambda$ be a fuzzy set in $(X, T)$. Since $\lambda \leq f^{-1}(f(\lambda))$ and $f(\lambda) \leq gfpcl(f(\lambda))$, it follows that $\lambda \leq f^{-1}(gfpcl(f(\lambda)))$. (8.2.6)

Now, $gfpcl(f(\lambda))$ is a gfpre-closed set in $(Y, S)$. Since $(Y, S)$ is a fuzzy pre-$T_{1/2}$ space, $gfpcl(f(\lambda))$ is fuzzy closed and hence fuzzy pre-closed in $(Y, S)$. Since $f$ is gfpre-continuous, $f^{-1}(gfpcl(f(\lambda)))$ is gfpre-closed. This implies,

$$gfpcl(f^{-1}(gfpcl(f(\lambda)))) = f^{-1}(gfpcl(f(\lambda))).$$

(8.2.7)

By (i) of Proposition 8.1.1 and by (8.2.6),

$$gfpcl(\lambda) \leq gfpcl(f^{-1}(gfpcl(f(\lambda)))).$$ (8.2.8)

By (8.2.7) and (8.2.8),

$$gfpcl(\lambda) \leq f^{-1}(gfpcl(f(\lambda))).$$
Therefore,

\[ f \left( \text{gfpc} \left( \lambda \right) \right) \leq \text{gfpc} \left( f \left( \lambda \right) \right). \tag{8.2.9} \]

Also by Proposition 8.2.2, \( \text{gfpc} \left( f \left( \lambda \right) \right) \leq f \left( \text{gfpc} \left( \lambda \right) \right). \tag{8.2.10} \]

Therefore, from (8.2.9) and (8.2.10),

\[ f \left( \text{gfpc} \left( \lambda \right) \right) = \text{gfpc} \left( f \left( \lambda \right) \right). \]

\((d) \Rightarrow (a)\) Let \( \lambda \) be a fuzzy set in \((X, T)\). By \((d)\), \( f \left( \text{gfpc} \left( \lambda \right) \right) = \text{gfpc} \left( f \left( \lambda \right) \right). \) Therefore, \( f \left( \text{gfpc} \left( \lambda \right) \right) \leq \text{gfpc} \left( f \left( \lambda \right) \right). \) Then by Proposition 8.2.1, \( f \) is fuzzy gc-pre-irresolute. Let \( \mu \) be any gfpre-closed set in \((X, T)\). Then,

\[ \text{gfpc} \left( \mu \right) = \mu, \text{ which implies that} \]

\[ f \left( \text{gfpc} \left( \mu \right) \right) = f \left( \mu \right). \]

By \((d)\), \( \text{gfpc} \left( f \left( \mu \right) \right) = f \left( \text{gfpc} \left( \mu \right) \right). \)

Therefore,

\[ \text{gfpc} \left( f \left( \mu \right) \right) = f \left( \mu \right), \text{ which implies that} f \left( \mu \right) \text{ is gfpre-closed in} (Y, S). \] Therefore, \( f^{-1} \) is fuzzy gc-pre-irresolute.

**Proposition 8.2.4**

Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \( f : (X, T) \to (Y, S) \) be fuzzy gc-pre-irresolute. Then \( \text{gfpo} \left( f^{-1} \left( \lambda \right) \right) \) is zero, for any gfpre-open set \( \lambda \) in \((Y, S)\).

**Proof**

Let \( \lambda \) be a gfpre-open set in \((Y, S)\). Then by hypothesis, \( f^{-1} \left( \lambda \right) \) is gfpre-open in \((X, T)\). Therefore,

\[ \text{gfpo} \left( f^{-1} \left( \lambda \right) \right) = f^{-1} \left( \lambda \right). \tag{8.2.6} \]
By definition, \( \text{gfpb} \left( f^{-1}(\lambda) \right) = f^{-1}(\lambda) - \text{gfpint} \left( f^{-1}(\lambda) \right) \).

Hence by (8.2.6),
\[
\text{gfpb} \left( f^{-1}(\lambda) \right) = f^{-1}(\lambda) - f^{-1}(\lambda) = 0.
\]

**8.3 INTERRELATIONS**

In this section, the concept of \( \text{g*fuzzy pre-} T_{1/2} \) space is introduced. In this regard, various interrelations among the concepts introduced in sections 8.1 and 8.2 are discussed.

**Proposition 8.3.1**

If \( \lambda \) is a gfpre-closed set then

(a) \( \text{gfpb} \left( \lambda \right) = \text{gfpFr} \left( \lambda \right) \).

(b) \( \text{gfpExt} \left( \lambda \right) = 1 - \lambda \).

**Proof**

(a) Since \( \lambda \) is gfpre-closed,
\[
\text{gfpcl} \left( \lambda \right) = \lambda.
\]

By definition, \( \text{gfpb} \left( \lambda \right) = \lambda - \text{gfpint} \left( \lambda \right) \)
\[
= \text{gfpcl} \left( \lambda \right) - \text{gfpint} \left( \lambda \right) \]
\[
= \text{gfpFr} \left( \lambda \right).
\]

Therefore, \( \text{gfpb}(\lambda) = \text{gfpFr} (\lambda) \).

(b) Since \( \lambda \) is gfpre-closed,
\[
\text{gfpcl} \left( \lambda \right) = \lambda.
\]

This implies, \( \text{gfpint} \left( 1 - \lambda \right) = 1 - \lambda \).

Therefore by definition,
\[
\text{gfpExt} \left( \lambda \right) = 1 - \lambda.
\]
**Proposition 8.3.2**

Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \(f : (X, T) \rightarrow (Y, S)\) be a fuzzy gfpre-continuous mapping. Then for any fuzzy pre-closed set \(\lambda\) in \((Y, S)\), \(\text{gfpc} (f^{-1}(\lambda)) = \text{gfpp} (f^{-1}(\lambda))\).

**Proof**

Let \(\lambda\) be a fuzzy pre-closed set in \((Y, S)\). Then by hypothesis, \(f^{-1}(\lambda)\) is gfpre-closed in \((X, T)\). Therefore,

\[\text{gfpc} (f^{-1}(\lambda)) = f^{-1}(\lambda).\]

Hence,

\[\text{gfpp} (f^{-1}(\lambda)) = \text{gfpc} (f^{-1}(\lambda)) - \text{gfpi} (f^{-1}(\lambda))\]

\[= f^{-1}(\lambda) - \text{gfpi} (f^{-1}(\lambda))\]

\[= \text{gfpc} (f^{-1}(\lambda)).\]

Therefore, \(\text{gfpp} (f^{-1}(\lambda)) = \text{gfpc} (f^{-1}(\lambda))\).

**Proposition 8.3.3**

Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \(f : (X, T) \rightarrow (Y, S)\) be any mapping. Then for any fuzzy set \(\lambda\) in \((Y, S)\), \(\text{gfpe} (f^{-1}(\lambda)) \leq \text{gfpc} (1 - f^{-1}(\lambda))\).

**Proof**

Let \(\lambda\) be any fuzzy set \(\lambda\) in \((Y, S)\). By definition,

\[\text{gfpe} (f^{-1}(\lambda)) = \text{gfpi} (1 - f^{-1}(\lambda))\]

\[\leq 1 - f^{-1}(\lambda),\]

by (a) of Proposition 8.1.1. Again by the same result,

\[\text{gfpe} (f^{-1}(\lambda)) \leq 1 - \text{gfpi} (f^{-1}(\lambda))\]

\[= \text{gfpc} (1 - f^{-1}(\lambda)).\]
Therefore, \( \text{gfpExt} \left( f^{-1}(\lambda) \right) \leq \text{gfpcl} \left( 1 - f^{-1}(\lambda) \right) \).

**Definition 8.3.1**

A fuzzy topological space \((X, T)\) is said to be a \(g^*\) fuzzy pre-\(T_{1/2}\) space (briefly, \(g^*f\) pre-\(T_{1/2}\)), if every gfp-pre-closed set in \((X, T)\) is generalized fuzzy closed (briefly, gf-closed) in \((X, T)\).

**Proposition 8.3.4**

Let \((X, T)\) be a \(g^*f\) pre-\(T_{1/2}\) space. For every gfp-pre-closed set \( \lambda \) in \((X, T)\), the following conditions hold:

(a) \( \text{gfb} \left( \lambda \right) = \text{gfFr} \left( \lambda \right) \) and

(b) \( \text{gfExt} \left( \lambda \right) = 1 - \lambda \).

**Proof**

(a) Let \( \lambda \) be a gfp-pre-closed set in \((X, T)\). Then by hypothesis, \( \lambda \) is gf-closed in \((X, T)\).

Therefore, \( \text{gfcl} \left( \lambda \right) = \lambda \). \hfill (8.3.1)

By definition, \( \text{gfb} \left( \lambda \right) = \lambda - \text{gfint} \left( \lambda \right) \)

\[= \text{gfcl} \left( \lambda \right) - \text{gfint} \left( \lambda \right), \text{ by } (8.3.1) \]

\[= \text{gfFr} \left( \lambda \right). \]

Hence, \( \text{gfb} \left( \lambda \right) = \text{gfFr} \left( \lambda \right) \).

(b) By definition,

\( \text{gfExt} \left( \lambda \right) = \text{gfint} \left( 1 - \lambda \right) \).

By (8.3.1), \( \text{gfint} \left( 1 - \lambda \right) = 1 - \lambda \).

Therefore, \( \text{gfExt} \left( \lambda \right) = 1 - \lambda \).

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Proposition 8.3.5

Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \(f : (X, T) \rightarrow (Y, S)\) be fuzzy gc-pre-irresolute and let \((X, T)\) be a g*f pre-\(T_{1/2}\) space. Then, for every gfpre-closed set \(\lambda\) in \((Y, S)\),

(a) \(gfb\ (f^{-1}(\lambda)) = gfFr\ (f^{-1}(\lambda))\) and

(b) \(gfExt\ (f^{-1}(\lambda)) = 1 - f^{-1}(\lambda)\).

Proof

(a) Let \(\lambda\) be a gfpre-closed set in \((Y, S)\). Then by hypothesis, \(f^{-1}(\lambda)\) is gfpre-closed in \((X, T)\). Since \((X, T)\) is a g*f pre-\(T_{1/2}\) space, \(f^{-1}(\lambda)\) is gf-closed in \((X, T)\). Therefore,

\[
gfcl\ (f^{-1}(\lambda)) = f^{-1}(\lambda).
\]  \hspace{1cm} \hspace{1cm} \text{(8.3.2)}

By definition,

\[
gfb\ (f^{-1}(\lambda)) = f^{-1}(\lambda) - gfint\ (f^{-1}(\lambda))
\]

\[
= gfcl\ (f^{-1}(\lambda)) - gfint\ (f^{-1}(\lambda))
\]

\[
= gfFr\ (f^{-1}(\lambda)), \text{ by (8.3.2)}. \hspace{1cm}
\]

Therefore,

\[
gfb\ (f^{-1}(\lambda)) = gfFr\ (f^{-1}(\lambda)).
\]

(b) By definition,

\[
gfExt\ (f^{-1}(\lambda)) = gfint\ (1 - f^{-1}(\lambda)).
\]

By (8.3.2), \(gfint\ (1 - f^{-1}(\lambda)) = 1 - f^{-1}(\lambda)\).

Therefore,

\[
gfExt\ (f^{-1}(\lambda)) = 1 - f^{-1}(\lambda).
\]

The above statement is not valid if \((X, T)\) is not a g*f pre-\(T_{1/2}\) space, as shown in the following Example.
Example 8.3.1

Let $X = \{a, b\}$. Define $T_1 = \{0,1,\lambda_1,\lambda_2,\lambda_3\}$, $T_2 = \{0,1,\mu_1,\mu_2\}$ where $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2 : X \to [0,1]$ are such that $\lambda_1(a) = 0.67, \lambda_1(b) = 0.64, \lambda_2(a) = 0.67, \lambda_2(b) = 0.35, \lambda_3(a) = 0.33, \lambda_3(b) = 0.34, \mu_1(a) = 0.75, \mu_1(b) = 0.67, \mu_2(a) = 0.67, \mu_2(b) = 0.49$. Clearly, $(X, T_1)$ and $(X, T_2)$ are fuzzy topological spaces.

Let $f : (X, T_1) \to (X, T_2)$ be the identity mapping. Let $\lambda : X \to [0,1]$ be any fuzzy set defined as $\lambda(a) = 0.33, \lambda(b) = 0.50$. Then, $\lambda$ is gfpre-closed in $(X, T_2)$ and $f^{-1}(\lambda)$ is gfpre-closed in $(X, T_1)$. **Therefore, $f$ is gc-pre-irresolute.** But $f^{-1}(\lambda) = \lambda$ is not gf-closed in $(X, T_1)$. **Hence, $(X, T_1)$ is not a $g^*f$ pre-$T_{1/2}$ space,** and the above statement is not valid.

Proposition 8.3.6

Let $f : (X, T)$ and $(Y, S)$ be any two fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ be a gfpre-closed mapping and let $(Y, S)$ be a $g^*f$ pre-$T_{1/2}$ space. For every gfpre-closed set $\lambda$ in $(X, T)$, the following statements hold:

(a) $gf\cap (f(\lambda)) = gf\text{Fr} (f(\lambda))$ and

(b) $gf\text{Ext} (f(\lambda)) = 1 - f(\lambda)$.

**Proof**

(a) Let $\lambda$ be a gfpre-closed set in $(X, T)$. Then by hypothesis, $f(\lambda)$ is gfpre-closed and hence gf-closed in $(Y, S)$. By definition,

$$gf\cap (f(\lambda)) = f(\lambda) - gf\text{Int} (f(\lambda))$$
\[ = \text{gfcl} (f(\lambda)) - \text{gfint} (f(\lambda)) \]
\[ = \text{gfFr} (f(\lambda)). \]

Therefore, \( fb(f(\lambda)) = \text{gfFr}(f(\lambda)). \)

(b) By definition,
\[ \text{gfExt}(f(\lambda)) = \text{gfint}(1 - f(\lambda)) \]
\[ = 1 - \text{gfcl}(f(\lambda)) \]
\[ = 1 - f(\lambda), \text{ since } f(\lambda) \text{ is gf-closed}. \]

Therefore, \( \text{gfExt}(f(\lambda)) = 1 - f(\lambda). \)

The above statement is not valid if \((Y, S)\) is not \(g^*f\text{-pre-T}_{1/2}\), as shown in the following Example.

Example 8.3.2

Let \(X = \{a, b\}\). Define \(T_1 = \{0, 1, \mu_1, \mu_2\}\), \(T_2 = \{0, 1, \lambda_1, \lambda_2, \lambda_3\}\) where \(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2\) are as in Example 8.3.1. Let \(f : (X, T_1) \rightarrow (X, T_2)\) be the identity mapping. Then, the fuzzy set \(\lambda\) which is as in Example 8.3.1 is gfpre-closed in \((X, T_1)\) and \(f(\lambda) = \lambda\) is gfpre-closed in \((X, T_2)\). Therefore, \(f\) is a gfpre-closed mapping. But \(f(\lambda)\) is not gf-closed in \((X, T_2)\). Hence, \((X, T_2)\) is not a \(g^*f\text{-pre-T}_{1/2}\) space and the above statement is not valid.

Proposition 8.3.7

Let \((X, T)\) be a \(g^*f\text{-pre-T}_{1/2}\) space; \((Y, S)\) and \((Z, R)\) be any two fuzzy topological spaces. Let \(f : (X, T) \rightarrow (Y, S)\) and \(g : (Y, S) \rightarrow (Z, R)\) be fuzzy gc-pre-irresolute mappings. For every gfpre-closed set in \((Z, R)\), the following statement hold:
(a) \( g f b \left( (g \circ f)^{-1}(\lambda) \right) = g f Fr \left( (g \circ f)^{-1}(\lambda) \right) \) and
(b) \( g f Ext \left( (g \circ f)^{-1}(\lambda) \right) = 1 - (g \circ f)^{-1}(\lambda) \).

**Proof**

(a) Let \( \lambda \) be any \( g f \) pre-closed set in \((Z, R)\). Then, since \( g \) is fuzzy gc pre-irresolute, \( g^{-1}(\lambda) \) is \( g f \) pre-closed in \((Y, S)\). And since \( f \) is fuzzy gc pre-irresolute, \( f^{-1}(g^{-1}(\lambda)) \) is \( g f \) pre-closed in \((X, T)\). That is, \( (g \circ f)^{-1}(\lambda) \) is \( g f \) pre-closed in \((X, T)\). Since \((X, T)\) is \( g^*f \) pre-\( T_{1/2} \), \( (g \circ f)^{-1}(\lambda) \) is \( g f \) closed in \((X, T)\). By definition, \( g f b \left( (g \circ f)^{-1}(\lambda) \right) = (g \circ f)^{-1}(\lambda) - g f int \left( (g \circ f)^{-1}(\lambda) \right) \)
\[
= g f c l \left( (g \circ f)^{-1}(\lambda) \right) - g f int \left( (g \circ f)^{-1}(\lambda) \right) \\
= g f Fr \left( (g \circ f)^{-1}(\lambda) \right).
\]
Therefore, \( g f b \left( (g \circ f)^{-1}(\lambda) \right) = g f Fr \left( (g \circ f)^{-1}(\lambda) \right) \).

(b) By definition, \( g f Ext \left( (g \circ f)^{-1}(\lambda) \right) \)
\[
= g f int \left( 1 - (g \circ f)^{-1}(\lambda) \right) \\
= 1 - g f c l \left( (g \circ f)^{-1}(\lambda) \right) \\
= 1 - (g \circ f)^{-1}(\lambda), \text{since } (g \circ f)^{-1}(\lambda) \text{ is gfclosed.}
\]
Therefore, \( g f Ext \left( (g \circ f)^{-1}(\lambda) \right) = 1 - (g \circ f)^{-1}(\lambda) \).

The above statement is not valid if \((X, T)\) is not \( g^*f \) pre-\( T_{1/2} \), as shown in the following Example.
Example 8.3.3

Let $X = \{ a, b \}$. Define $T_1 = \{ 0, 1, \lambda_1, \lambda_2, \lambda_3 \}$, $T_2 = \{ 0, 1, \mu_1, \mu_2 \}$, $T_3 = \{ 0, 1, \delta_1, \delta_2 \}$ where $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \delta_1, \delta_2 : X \rightarrow [0, 1]$ are such that $\lambda_1 (a) = 0.67, \lambda_1 (b) = 0.64, \lambda_2 (a) = 0.67, \lambda_2 (b) = 0.35, \lambda_3 (a) = 0.33, \lambda_3 (b) = 0.34, \mu_1 (a) = 0.75, \mu_1 (b) = 0.67, \mu_2 (a) = 0.67, \mu_2 (b) = 0.49, \delta_1 (a) = 0.75, \delta_1 (b) = 0.75, \delta_2 (a) = 0.67, \delta_2 (b) = 0.4$. Let $f : (X, T_1) \rightarrow (X, T_2)$ and $g : (X, T_2) \rightarrow (X, T_3)$ be the identity mappings. Let $\lambda : X \rightarrow [0, 1]$ be any fuzzy set defined as $\lambda (a) = 0.33, \lambda (b) = 0.50$. Then, $\lambda$ is gfpre-closed in $(X, T_3)$. Since $g^{-1}(\lambda)$ is gfpre-closed in $(X, T_2)$ and $(g \circ f)^{-1}(\lambda)$ gfpre-closed in $(X, T_1)$, $f$ and $g$ are fuzzy gc-pre-irresolutes. But $(X, T_1)$ is not gfpre-T_{1/2} as $(g \circ f)^{-1}(\lambda)$ is gfpre-closed but not gf-closed in $(X, T_1)$. Hence, the above statement is not valid.