CHAPTER 3

INTERVAL-VALUED FUZZY REGULAR LANGUAGE

3.1 INTRODUCTION

In this chapter, using the concept of interval-valued fuzzy sets [16], one of the extensions of fuzzy sets, an attempt has been made to generalize the membership function of fuzzy language. The language so obtained is termed as an interval-valued fuzzy language. Here, the membership value of each string is a closed subinterval in [0, 1]. This membership function approximates the correct (but unknown) membership degree of each string in the given language. If each string of such language is regular with finite membership value, then the language is said to be an interval-valued fuzzy regular language. The concept of interval-valued fuzzy language provides us with a flexible mathematical framework to cope with imprecise information. This language has been studied through proposed finite automata (DFA and NDFA) with interval-valued fuzzy transitions and also through finite automata (DFA and NDFA) with interval-valued fuzzy final states via some theorems. The transition of proposed automata possesses upper and lower membership values in the unit interval. By ignoring the upper membership value, these will act like conventional fuzzy automata. Here, the pessimistic part of the extended fuzzy automata has been discussed.

An interval-valued fuzzy regular language can be described by interval-valued fuzzy regular expression. Minimization of fuzzy automata through Myhill-Nerode theorem [66] has been extended to interval-valued fuzzy (final) states automata which facilitate to reduce the number of redundant states in interval-valued fuzzy (final) states. Again an algorithm is proposed to find minimized interval-valued fuzzy (final) states.

The main contributions of this chapter are fourfold. First, we have proposed the definition of interval-valued fuzzy language and interval-valued fuzzy regular language and studied their related properties. Second, we have discussed the acceptance of interval-valued fuzzy regular language through finite automata (DFA and NDFA) with interval-valued fuzzy transitions and interval-valued fuzzy (final) states. Thirdly, we have proposed the definition of interval-valued fuzzy regular expressions. Finally, Myhill-Nerode theorem has been extended to interval-valued
fuzzy regular language and an algorithm is proposed for minimization of interval-valued fuzzy (final) states automaton.

### 3.2 INTERVAL-VALUED FUZZY REGULAR LANGUAGE

**Definition 3.2.1:** Let $\Sigma$ be an alphabet. Then, we call the set

$$\tilde{L} = \{(w, [f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w)]) | w \in \Sigma^*\}$$

an *interval-valued fuzzy language* (IVFL), where

$$f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w) : \Sigma^* \to [0, 1]$$

represents respectively, the lower and upper membership functions of $\tilde{L}$ such that $0 \leq f^L_{\tilde{L}}(w) \leq f^U_{\tilde{L}}(w) \leq 1, \forall w \in \Sigma^*$.

In short,

$$\tilde{L} = \{(w, f_{\tilde{L}}(w)) | w \in \Sigma^*\}, \text{ where } f_{\tilde{L}}(w) = [f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w)], \forall w \in \Sigma^*.$$

In particular, $f_{\tilde{L}}(w) : \Sigma^* \to I[0, 1]$ denotes the membership function of $\tilde{L}$.

Here, $f_{\tilde{L}}(w) \in I[0, 1]$ and not a real number in $[0, 1]$ which is assigned to each string of the language $\tilde{L}$.

**Example 3.1:** If $\Sigma = \{a, b\}$ and $f_{\tilde{L}} : \Sigma^* \to I[0, 1]$ then,

$$\tilde{L} = \{(a^*, [0.3, 0.5]), (a^* b, [0.4, 0.7]) | a^*, a^* b \in \Sigma^*\}$$

represents an IVFL.

Let $\tilde{L}$ be an IVFL over an alphabet $\Sigma$ and $f_{\tilde{L}} : \Sigma^* \to I[0, 1]$ the membership function of $\tilde{L}$. Then, for each $[m, n] \in I[0, 1]$ denote by $S_{\tilde{L}}[m, n]$ the set

$$S_{\tilde{L}}[m, n] = \{w | w \in \Sigma^* \& f_{\tilde{L}}(w) = [m, n]\}.$$ Note that $S_{\tilde{L}}$ as a function is just $f_{\tilde{L}}^{-1}$.

We consider, interval-valued fuzzy grammar as the generator of IVFL and is defined below.

**Definition 3.2.2:** An *interval-valued fuzzy grammar* (IVFG) is a quadruple $G = (V_N, V_T, P, S)$, where

- a) $V_T$ is the finite set of alphabets called terminal symbols,
- b) $V_N$ is the finite set of non-terminal symbols ($V_N \cap V_T = \emptyset$),
- c) $P$ is a list of production (rewrite) rules, and
d) $S$ is an initial symbol included in $V_N$.

The elements of $P$ are expressions of the form

$$
\mu(\alpha \rightarrow \beta) = [\chi, \psi], \text{ for } \chi, \psi \in [0,1] \text{ and } 0 \leq \chi + \psi \leq 1, \text{ where } \alpha \text{ and } \beta \text{ are strings in } (V_N \cup V_T)^* \text{ and } \chi, \psi \text{ represents respectively, the lower membership and upper membership of } \beta \text{ for } a \text{ given } \alpha \text{ such that } \chi \leq \psi.
$$

For convenience we abbreviate

$$
\mu(\alpha \rightarrow \beta) = [\chi, \psi] \text{ to } \alpha \rightarrow^{|\chi, \psi|} \beta \text{ or, more simply, } \alpha \rightarrow \beta.
$$

Interval-valued fuzzy regular grammar (IVFRG) allows the productions of the form

$$
A \rightarrow^{|\chi, \psi|} aB \text{ or } A \rightarrow^{|\chi, \psi|} a, A, B \in V_N, a \in V_T, 0 \leq \chi, \psi \leq 1.
$$

In addition, the production $S \rightarrow \varepsilon$ is allowed with the lower membership and upper membership values 1 & 1 respectively.

**Some Algebraic Operations over IVFLs**

Let $\tilde{L}_1$ and $\tilde{L}_2$ be two IVFLs over an alphabet $\Sigma$ with the membership functions $f_{\tilde{L}_1}$ and $g_{\tilde{L}_2}$ respectively. Then, the basic operations such as union, intersection, complement, concatenation and star operations on $\tilde{L}_1$ and $\tilde{L}_2$ can be defined in the following way.

1. **Union:** The union of $\tilde{L}_1$ and $\tilde{L}_2$ is defined by

$$
\tilde{L} = \tilde{L}_1 \cup \tilde{L}_2 = \{(w, \max\{f_{\tilde{L}_1}^L(w), g_{\tilde{L}_2}^L(w)\}, \max\{f_{\tilde{L}_1}^U(w), g_{\tilde{L}_2}^U(w)\}) \mid w \in \Sigma^*\}.
$$

2. **Intersection:** The intersection of $\tilde{L}_1$ and $\tilde{L}_2$ is defined by

$$
\tilde{L} = \tilde{L}_1 \cap \tilde{L}_2 = \{(w, \min\{f_{\tilde{L}_1}^L(w), g_{\tilde{L}_2}^L(w)\}, \min\{f_{\tilde{L}_1}^U(w), g_{\tilde{L}_2}^U(w)\}) \mid w \in \Sigma^*\}.
$$

3. **Complement:** The complement of $\tilde{L}_1$ is defined by

$$
\tilde{L} = \tilde{L}_1^c = \{(w, [1 - f_{\tilde{L}_1}^L(w), 1 - f_{\tilde{L}_1}^L(w))] \mid w \in \Sigma^*\}.
$$

4. **Concatenation:** The concatenation of $\tilde{L}_1$ and $\tilde{L}_2$ is defined by

$$
\tilde{L} = \tilde{L}_1 \cdot \tilde{L}_2 = \{(w, \max\{\min\{f_{\tilde{L}_1}^L(x), g_{\tilde{L}_2}^L(y)\}, \max\{f_{\tilde{L}_1}^U(x), g_{\tilde{L}_2}^U(y)\}\}) \mid w = xy, x, y \in \Sigma^*, w \in \Sigma^*\}.
$$

5. **Star:** The star operation on $\tilde{L}_1$ is defined by
\( \tilde{L} = \tilde{L}_4 = \{ (w, [\max \{ \min (f_{I_4}^L(x_1), f_{I_4}^L(x_2), \ldots, f_{I_4}^L(x_n)) \}], \max \{ \min (f_{I_4}^U(x_1), f_{I_4}^U(x_2), \ldots, f_{I_4}^U(x_n)) \}] \mid w = x_1 x_2 \ldots x_n, x_1, x_2, \ldots, x_n \in \sum^*, n \geq 0 \} \)

assuming that \( \min \varepsilon = [0, 0] \), where \( \varepsilon \) being the empty string.

Hence, we have \( \tilde{L}^* = \bigcup_{i=0}^{\infty} \tilde{L}^i = \tilde{L}^0 \cup \tilde{L}^1 \cup \tilde{L}^2 \cup \ldots \).

i.e., Kleene closure is satisfied.

6. The + operation on \( \tilde{L}_1 \) is defined by
\[
\tilde{L} = \tilde{L}_1^+ = \{ (w, [\max \{ \min (f_{I_4}^L(x_1), f_{I_4}^L(x_2), \ldots, f_{I_4}^L(x_n)) \}], \max \{ \min (f_{I_4}^U(x_1), f_{I_4}^U(x_2), \ldots, f_{I_4}^U(x_n)) \}] \mid w = x_1 x_2 \ldots x_n, x_1, x_2, \ldots, x_n \in \sum^*, n \geq 1 \}.
\]

Hence, we have \( \tilde{L}^* = \bigcup_{i=1}^{\infty} \tilde{L}^i = \tilde{L}^1 \cup \tilde{L}^2 \cup \ldots \). i.e., Positive closure is satisfied.

We consider, IVFLs as a special class of interval-valued fuzzy sets so, the equivalence and inclusion relation between two IVFLs are the equivalence and inclusion relation between two interval-valued fuzzy sets. i.e., \( \tilde{L}_1 \subset \tilde{L}_2 \) iff \( f_{I_4}(w) = g_{I_2}(w), \forall w \in \sum^* \), and
\[ \tilde{L}_1 \subseteq \tilde{L}_2 \text{ iff } f_{I_4}(w) \subseteq g_{I_2}(w), \forall w \in \sum^* \), where \( \tilde{L}_1 \) and \( \tilde{L}_2 \) are two IVFLs over \( \sum \).

**Definition 3.2.3:** Let \( \tilde{L} \) be an IVFL over an alphabet \( \sum \) and \( f_{I}(w) : \sum^* \to I[0, 1] \) the membership function of \( \tilde{L} \). Then, we call \( \tilde{L} \) an interval-valued fuzzy regular language (IVFRL) if:

i. the set \( \{ [m, n] \mid [m, n] \in I[0, 1] \& S_{I}[m, n] \neq \emptyset \} \) is finite, and

ii. for each \( [m, n] \in I[0, 1] \), the set \( S_{I}[m, n] \) is regular.

**Theorem 3.1:** IVFRLs are closed under union, intersection, complement, concatenation and star operations.

**Proof:** Let \( \tilde{L}_1 \) and \( \tilde{L}_2 \) be two IVFRLs over an alphabet \( \sum \).

Let \( f_{I_4} : \sum^* \to M \) \& \( g_{I_2} : \sum^* \to N \) be the membership function of \( \tilde{L}_1 \) and \( \tilde{L}_2 \) respectively, where \( M \) and \( N \) represents the set of all closed subintervals in \([0, 1]\). Obviously, \( O \subseteq M \cup N \) (in the case of union, intersection, concatenation or star operation) and

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\[ O = \{[1 - f^U_{L_i}, 1 - f^L_{L_i}] | [f^L_{L_i}, f^U_{L_i}] \in M \} \] (in the case of complementation) are finite and corresponding strings are regular.

Note that, \( O \) represents the membership function of new IVFRL obtained after an operation (union, intersection, complement, concatenation or star). It is described as follows.

1. **Union:**

\[
S^L_{L}[m,n] = \begin{cases} 
S^L_{L_1}[m,n] - \cup_{[m,n]} \cup_{[m,n]} S^L_{L_2}[m,n], & \text{if } [m,n] \in M - N, \\
S^L_{L_2}[m,n] - \cup_{[m,n]} \cup_{[m,n]} S^L_{L_1}[m,n], & \text{if } [m,n] \in N - M, \\
((S^L_{L_1}[m,n] \cup S^L_{L_2}[m,n]) - \cup_{[m,n]} \cup_{[m,n]} S^L_{L_4}[m,n]) - \\
\cup_{[m,n]} \cup_{[m,n]} S^L_{L_2}[m,n], & \text{if } [m,n] \in M \cap N.
\end{cases}
\]

2. **Intersection:**

\[
S^L_{L}[m,n] = \begin{cases} 
S^L_{L_1}[m,n] - \cup_{[m,n]} S^L_{L_2}[m,n], & \text{if } [m,n] \in M - N, \\
S^L_{L_2}[m,n] - \cup_{[m,n]} S^L_{L_1}[m,n], & \text{if } [m,n] \in N - M, \\
((S^L_{L_1}[m,n] \cap S^L_{L_2}[m,n]) - \cup_{[m,n]} S^L_{L_4}[m,n]) - \\
\cup_{[m,n]} S^L_{L_2}[m,n], & \text{if } [m,n] \in M \cap N.
\end{cases}
\]

3. **Complement:** \( O = \{([1 - n, 1 - m]) | [m,n] \in M \}, S^L_{L}[m,n] = S^L_{L_4}[1 - n, 1 - m]. \)

4. **Concatenation:**

\[
S^L_{L}[m,n] = \bigcup_{m_2, n_2 \in N} \bigcup_{m_1, n_1 \in M \cap N} S^L_{L_1}[m_1, n_1] S^L_{L_2}[m_2, n_2] - \\
\bigcup_{m_2, n_2 \in N} \bigcup_{m_1, n_1 \in M \cap N} S^L_{L_4}[m_1, n_1] S^L_{L_2}[m_2, n_2].
\]

5. **Star:**

Assuming that \( M = \{[m_1, n_1], [m_2, n_2], ..., [m_l, n_l]\} \) and \([1,1] \ge [m_1, n_1] > [m_2, n_2] > ... > [m_l, n_l] \ge [0,0] \).

\[
S^L_{L}[m_1, n_1] = (S^L_{L_4}[m_1, n_1])^*, \text{ if } [m_1, n_1] = [1,1], S^L_{L}[1,1] = \varepsilon \text{ and}
\]

\[
S^L_{L}[m_1, n_1] = (S^L_{L_4}[m_1, n_1])^+ - \varepsilon, \text{ if } [m_1, n_1] \neq [1,1].
\]

\[
S^L_{L}[m_i, n_i] = (\bigcup_{j \le l} S^L_{L_4}[m_j, n_j])^+ - (\bigcup_{k<i} S^L_{L}[m_k, n_k]) - \varepsilon, \text{ if } 1 < i \le l.
\]

Hence, Kleene closure is satisfied.
3.3 FINITE AUTOMATA WITH INTERVAL-VALUED FUZZY TRANSITIONS

Definition 3.3.1: A nondeterministic finite automaton with interval-valued fuzzy transitions (NDFA-IVFT) $A$ is a 5-tuple $A = (Q, \Sigma, \delta, S, F)$, where $Q$ is the set of finite states, $\Sigma$ is the set of finite input alphabets, $\delta : Q \times \Sigma \times Q \rightarrow I[0, 1]$ is the membership function of state transition, $S$ is the starting state and $F \subseteq Q$ is the set of final states.

For $x \in \Sigma^*$ and $p, q \in Q$ define, the extended membership function of state transition $\delta$ as:

$$
\delta^*(p, x, q) = \begin{cases} 
[0, 1], & \text{if } x = \varepsilon \text{ and } p \neq q, \\
[0, 0], & \text{if } x = \varepsilon \text{ and } p = q, \\
\max_{r \in Q} \{\min(\delta^*(p, x', r), \delta(r, a, q)) | x' \in \Sigma^*, a \in \Sigma\}, & \text{otherwise}.
\end{cases}
$$

Hence, we say that the string $x \in \Sigma^*$ is accepted by $A$ with the degree $d_A(x)$, where $d_A(x) = \max\{\delta^*(S, x, q) | q \in F\}$. We denote the language accepted by $A$ by $\tilde{L}(A)$ and is given by the set $\tilde{L}(A) = \{(x, d_A(x)) | x \in \Sigma^*\}$.

Example 3.2: Let $\tilde{A} = (Q, \Sigma, \tilde{\delta}, S, F, \tilde{\delta})$ be a NDFA-IVFT as shown in the figure below, where $Q = \{S, q_1, q_2\}$, $\Sigma = \{a, b\}$, $S = S$, $F = \{q_2\}$, $\tilde{\delta}(S, a, q_1) = [0.3, 0.8]$, $\tilde{\delta}(S, a, q_2) = [0.3, 0.8]$, $\tilde{\delta}(q_1, b, q_1) = [0.7, 0.9]$, $\tilde{\delta}(q_1, a, q_2) = [0.6, 0.7]$, $\tilde{\delta}(q_2, b, q_2) = [0.5, 0.8]$.

![NDFA-IVFT](image)

Figure 3.1: NDFA-IVFT

The interval-valued fuzzy regular language accepted by the above NDFA-IVFT is:

$$
\tilde{L}(\tilde{A}) = \{(x, [0.6, 0.7]) | x \in ab^*ab^*\} \cup \{(y, [0.5, 0.8]) | y \in ab^*\}.
$$
Definition 3.3.2: A deterministic finite automaton with interval-valued fuzzy transitions (DFA-IVFT) is a NDFA-IVFT with the condition that for each \( p, q \in Q \) and \( a \in \Sigma \), if \( \delta(p, a, q) > [0, 0] \) and \( \delta(p, a, q') > [0, 0] \) then \( q = q' \).

Theorem 3.2: ‘\( \tilde{L} \)’ is an IVFRL if and only if ‘\( L \)’ is accepted by a NDFA-IVFT ‘\( A \)’ with the exception of \( \varepsilon \) the empty string.

Proof: Let ‘\( \tilde{L} \)’ be an IVFRL and \( f_\tilde{L} : \Sigma^* \rightarrow M \) be the membership function of ‘\( \tilde{L} \)’, where \( M \) is the set of all closed subintervals in \([0, 1]\).

Let \( M = \{m_1, m_2, \ldots, m_n\} \) for some \( n \geq 1 \). Then, \( S_\tilde{L}(m_i) \) is regular for each \( m_i \in M, 1 \leq i \leq n \).

Note that, \( S_\tilde{L}(m_i) \cap S_\tilde{L}(m_j) = \emptyset \) for \( i \neq j \) since, \( f_\tilde{L} \) is a function.

Let \( A_i = (Q_i, \Sigma, \delta_i, S_i, F_i) \) be a DFA-IVFT (or a NDFA-IVFT) such that \( S_\tilde{L}(m_i) = \tilde{L}(A_i), 1 \leq i \leq n \). We construct, \( A' = (Q, \Sigma, \tilde{\delta}, S, F) \), where

\[
\tilde{\delta}_i(p, a, q) = \begin{cases} m_i, & \text{if } (p, a, q) \in \delta_i, \\ [0,0], & \text{otherwise.} \end{cases}
\]

We assume that \( Q_i \cap Q_j = \emptyset \) for \( i \neq j \). Define, \( A = (Q, \Sigma, \delta, S, F) \) such that

\[
Q = Q_1 \cup Q_2 \cup \ldots \cup Q_n \cup \{S\}, S \notin Q_1 \cup Q_2 \cup \ldots \cup Q_n, F = F_1 \cup F_2 \cup \ldots \cup F_n,
\]

\[
\delta(p, a, q) = \begin{cases} \delta_i(p, a, q), & \text{if } p, q \in Q_i, 1 \leq i \leq n, \\ \delta_S(p, a, q), & \text{if } p = S, q \in Q_i, 1 \leq i \leq n, \\ [0,0], & \text{otherwise.} \end{cases}
\]

It is clear that, ‘\( A \)’ accepts ‘\( \tilde{L} \)’ with the exception of \( \varepsilon \) the empty string.

Conversely,

let \( A = (Q, \Sigma, \delta, S, F) \) be a NDFA-IVFT.

Define an IVFL ‘\( \tilde{L} \)’ with \( f_\tilde{L}(w) = d_A(w) \) for each \( w \in \Sigma^* \) (here \( f_\tilde{L}(\varepsilon) = [0, 0] \)).

Now we have to show that ‘\( \tilde{L} \)’ is regular.

Let \( M = \{[m, n] \mid \delta(p, a, q) = [m, n] \text{ for some } p, q \in Q, a \in \Sigma\} \).

Obviously, \( M \) is finite.

Now assume that \( M = \{m_1, m_2, \ldots, m_n\} \) with \( m_1 > m_2 > \ldots > m_n \), \( n \geq 1 \) for each \( i, 1 \leq i \leq n \),

define, a NDFA-IVFT, \( A_i = (Q_i, \Sigma, \delta_i, S_i, F_i) \), where \( \delta_i = \{(p, a, q) \mid \delta(p, a, q) \geq m_i\} \).

Define the languages \( \tilde{L}_i, 1 \leq i \leq n \), as shown below in the increasing sequence of \( i \):
\[ \tilde{L}_i = \tilde{L}(A_i), \]
\[ \tilde{L}_2 = \tilde{L}(A_2) - \tilde{L}(A_1), \]
\[ \tilde{L}_3 = \tilde{L}(A_3) - (\tilde{L}(A_2) \cup \tilde{L}(A_1)), \]
\[ \vdots \]
\[ \tilde{L}_n = \tilde{L}(A_n) - \bigcup_{j=1}^{i-1} \tilde{L}(A_j). \]

Then, \( S_L(m_i) = \tilde{L}_i \) and \( \tilde{L}_i \) is regular for each \( i, 1 \leq i \leq n \).

Hence, \( \tilde{L} \) is an IVFRL.

**Theorem 3.3:** Let \( \tilde{L} \) be an IVFRL. Then \( \tilde{L} \) is accepted by a DFA-IVFT iff it satisfies the following condition: For \( x, y \in \Sigma^+, u \in \Sigma^* \)

1. \( x = yu \) and \( f_{\tilde{L}}(y) \geq [0, 0] \) implies that \( f_{\tilde{L}}(x) \leq f_{\tilde{L}}(y) \).

**Proof:** Let \( \tilde{L} \) be accepted by a DFA-IVFT \( A = (Q, \Sigma, \delta, S, F) \).

We show that \( \tilde{L} \) satisfies the given condition (i.e., (1)).

Let \( x = yu \), for \( x, y \in \Sigma^+ \) and \( u \in \Sigma^* \).

If \( d_A(x) = 0 \) then, \( f_{\tilde{L}}(x) \leq f_{\tilde{L}}(y) \) is true. Otherwise,

\[ f_{\tilde{L}}(x) = d_A(x) = \min\{\delta^*(S, y, q), \delta^*(q, u, f)\} \leq \delta^*(S, y, q) = d_A(y) = f_{\tilde{L}}(y), \]

where \( q, f \in F \).

Conversely,

let \( \tilde{L} \) be an IVFRL, with \( f_{\tilde{L}} : \Sigma^* \rightarrow M \) as membership function, where \( M \) is the set of all closed subintervals in \([0, 1]\) satisfying the given condition.

Assume that \( M = \{m_1, m_2, \ldots, m_n\} \) for some \( n \geq 1 \).

It is clear from the theorem 3.2 that, we can construct a DFA,

\( A_i = (Q_i, \Sigma, \delta_i, S_i, F_i) \) such that \( \tilde{L}(A_i) = S_L(m_i) \) for each \( i, 1 \leq i \leq n \).

Note that for \( 1 \leq i, j \leq n \) and \( i \neq j \), \( \tilde{L}(A_i) \cap \tilde{L}(A_j) = S_L(m_i) \cap S_L(m_j) = \phi \).

Now we construct a DFA, \( A = (Q, \Sigma, \delta, S, F) \), where \( Q = Q_1 \times Q_2 \times \ldots \times Q_n, S = (S_1, S_2, \ldots, S_n), \delta : Q \times \Sigma \rightarrow Q \) is defined by,
\[\delta((q_1, q_2, \ldots, q_n), a) = (\delta_1(q_1, a), \delta_2(q_2, a), \ldots, \delta_n(q_n, a))\]

\[F = F'_1 \cup F'_2 \cup \ldots \cup F'_n, \text{ where}
F'_i = \{(q_1, q_2, \ldots, q_n) \mid (q_1, q_2, \ldots, q_n) \in Q, q_i \in F_i \text{ and } q_j \notin F_j \text{ for } i \neq j, 1 \leq i, j \leq n\}.

From this it is clear that \(F'_i \cap F'_j = \emptyset\) for \(i \neq j\) and \(F'_i = \{w \mid w \in \Sigma^* \text{ and } \delta^*(S, w) \in F'_i\}\).

Based on the above DFA ‘A’, we can define a DFA-IVFT
\[\tilde{A} = (Q, \Sigma, \tilde{\delta}, S, F),\]

such that
\[\tilde{\delta}(p, a, q) = \begin{cases} m_i, & \text{if } \delta(p, a) = q \in F'_i, \\ [1, 1], & \text{if } \delta(p, a) = q \notin F, \\ [0, 0], & \text{otherwise.} \end{cases}\]

It remains to show that \(d_{\tilde{A}}(w) = f_{\tilde{L}}(w)\), for each \(w \in \Sigma^\star\).

But first we show that \(\tilde{A}\) has the following property:

For each \(w \in \Sigma^\star\) with \(w = xa \forall x \in \Sigma^\ast, a \in \Sigma\).

\[d_{\tilde{A}}(w) = m_i > [0, 0] \text{ iff } \delta^*(S, x, p) \geq m_i \text{ and } \delta(p, a, q) = m_j \text{ for some } q \in F_i, 1 \leq i \leq n. \text{ ---- (B)}\]

If part is obvious.

For the only if part, it holds trivially when \(x = \varepsilon\).

For \(x \neq \varepsilon\), we assume the contrary, i.e.,
\[\delta^*(S, x, q) = m_i \text{ and } \delta(p, a, q) = m_j > m_i \text{ for } 1 \leq i, j \leq l.\]

Then, there exists a decomposition of \(x = ybz\), where \(y, z \in \Sigma^\ast\) and \(b \in \Sigma\), such that
\[\delta^*(S, y, r) \geq m_i, \delta(r, b, t) = m_i \text{ and } \delta(t, z, p) \geq m_i.\]

By the definition of ‘\(\tilde{A}\)’ we know that \(t \in F_i\) and \(q \in F_j\).

Thus, we have \(f_{\tilde{L}}(yb) = m_i\) and \(f_{\tilde{L}}(w) = m_j\).

Since we assume that \(m_j > m_i\), this is a contradiction to the specified condition (i.e., (1)).

So (B) holds.

Furthermore, (B) implies that, \(xa \in S'_L(m_j)\) i.e., \(f_{\tilde{L}}(w) = m_i\).

This completes the proof.
3.4 FINITE AUTOMATA WITH INTERVAL-VALUED FUZZY (FINAL) STATES

In this section, the definition of finite automaton with interval-valued fuzzy (final) states (NDFA-IVFS and DFA-IVFS) is proposed. IVFRL is accepted by NDFA-IVFS and DFA-IVFS without any restrictions and vice versa. Hence, to study IVFRL these models are more suitable.

Definition 3.4.1: A nondeterministic finite automaton with interval-valued fuzzy (final) states (NDFA-IVFS) is a 5-tuple $\tilde{A}=(Q, \Sigma, \tilde{\delta}, S, F_{\tilde{A}})$, where $Q$ is the finite set of states, $\Sigma$ is the finite set of input alphabets, $\tilde{\delta} : Q \times \Sigma \rightarrow 2^{\emptyset}$ is the transition function, $S$ is the starting state and $F_{\tilde{A}} : Q \rightarrow I[0, 1]$ the membership function of interval-valued fuzzy (final) state set.

Define,
\[ d_{\tilde{A}}(x) = \max\{ F_{\tilde{A}}(q) \mid (S, x, q) \in \tilde{\delta}^* \}, \]
where $\tilde{\delta}^*$ is the reflexive and transitive closure of $\tilde{\delta}$. Then, we say that the string ‘$x$’ is accepted by $\tilde{A}$ with the degree $d_{\tilde{A}}(x)$. So, the interval-valued fuzzy regular language accepted by $\tilde{A}$ i.e., $\tilde{L}(\tilde{A})$ is given by the set, $\tilde{L}(\tilde{A}) = \{(x, d_{\tilde{A}}(x)) \mid x \in \Sigma^* \}$.

Example 3.3:

Let $\tilde{A}=(Q, \Sigma, \tilde{\delta}, S, F_{\tilde{A}})$ be a NDFA-IVFS (Figure 3.2) with $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{d, o\}$, $q_1 = S$ the interval-valued fuzzy starting state with membership value $F_{\tilde{A}}(q_1) = [0.3, 0.5]$, $\tilde{\delta} : Q \times \Sigma \rightarrow 2^\emptyset$ the transition function given as
\[ \tilde{\delta}(q_1, d) = q_1, \tilde{\delta}(q_1, o) = q_2, \text{and} \ F_{\tilde{A}} : Q \rightarrow M, \] where $M = I[0, 1]$

the membership function of interval-valued fuzzy (final) state set given as $F_{\tilde{A}}(q_1) = [0.3, 0.5], F_{\tilde{A}}(q_2) = [0.4, 0.7], F_{\tilde{A}}(q_3) = [0.6, 0.9]$. Then, $d_{\tilde{A}}(do) = [0.6, 0.9], d_{\tilde{A}}(d) = [0.4, 0.7], \text{and} \ d_{\tilde{A}}(dd^+) = [0.4, 0.7]$. 
Definition 3.4.2: A deterministic finite automaton with interval-valued fuzzy (final) state (DFA-IVFS) \( \tilde{A} = (Q, \Sigma, \tilde{\delta}, S, F_{\tilde{A}}) \) is a NDFA-IVFS with \( \tilde{\delta} : Q \times \Sigma \rightarrow Q \) as function.

Thus, for each \( x \in \Sigma^* \), \( d_{\tilde{A}}(x) = F_{\tilde{A}}(q) \), where \( q = \tilde{\delta}^*(S, x) \).

Define, \( d_{\tilde{A}}(x) = [0, 0] \) if \( \tilde{\delta}^*(S, x) \) is not defined.

Theorem 3.4: Let \( \tilde{L} \) be an IVFL. Then \( \tilde{L} \) is an IVFRL iff it is accepted by a DFA-IVFS.

Proof: Let \( \tilde{L} \) be an IVFL with the membership function \( f_{\tilde{L}} : \Sigma^* \rightarrow M \), where \( M \) is the set of all closed subintervals in \([0, 1]\).

Assume that \( \tilde{L} \) is an IVFRL.

Then, \( M \) is finite and for each, \( m \in M \) the set \( S_{\tilde{L}}(m) \) is regular.

Assume that \( M = \{m_1, m_2, \ldots, m_n\} \). We construct a DFA,
\( \tilde{A}_i = (Q_i, \Sigma, \tilde{\delta_i}, S_i, F_{\tilde{A}_i}) \) for each \( i, 1 \leq i \leq n \) such that \( \tilde{L}(A_i) = S_{\tilde{L}}(m_i) \).

Define a DFA-IVFS \( \tilde{A} = (Q, \Sigma, \tilde{\delta}, S, F_{\tilde{A}}) \) to be the cross product of \( \tilde{A}_i, 1 \leq i \leq n \) with
\[
F_{\tilde{A}}(q^{(1)}, q^{(2)}, \ldots, q^{(n)}) = \begin{cases} 
m_i, & \text{if } q^{(i)} \in F_{\tilde{A}_i} \text{ for some } i, 1 \leq i \leq n, q^{(j)} \notin F_j, \forall j \neq i, \\
[0,0], & \text{otherwise}. 
\end{cases}
\]

Note that \( (q^{(1)}, q^{(2)}, \ldots, q^{(n)}) \) is reachable from \( (S_1, S_2, \ldots, S_n) \) in \( \tilde{A} \),
then it is not possible to get
\( q^{(i)} \in F_{\tilde{A}_i} \) and \( q^{(j)} \in F_{\tilde{A}_j} \) for \( i \neq j \), since \( \tilde{L}(A_i) \cap \tilde{L}(A_j) = \phi \) for \( i \neq j, 1 \leq i, j \leq n \).

Hence, \( \tilde{A} \) accepts \( \tilde{L} \).
Conversely, let \( \tilde{A} = (Q, \Sigma, \tilde{\delta}, S, F_{\tilde{A}}) \) be a DFA-IVFS.

Define, \( M = \{ m \mid F_{\tilde{A}}(q) = m \text{ for some } q \in Q \} \). So, \( M \) is finite.

For each \( m \in M \) define, \( \tilde{A}_m = (Q, \Sigma, \tilde{\delta}, S, F_{\tilde{A}_m}) \), where \( F_{\tilde{A}_m} = \{ q \mid F_{\tilde{A}}(q) = m \} \).

Let \( \tilde{L} = \tilde{L}(\tilde{A}) \) i.e., \( f_\tilde{L}(w) = d_\tilde{A}(w) \).

Then clearly, for each \( m \in M \), \( S_\tilde{L}(m) = \tilde{L}(\tilde{A}_m) \) is regular.

Thus, \( \tilde{L} \) is an IVFRL.

**Theorem 3.5:** An IVFL is accepted by a NDFA-IVFS iff it is accepted by a DFA-IVFS.

**Proof:** Let ‘\( \tilde{L} \)’ be an IVFL. Here we have to show if \( \tilde{A} \) is a NDFA-IVFS and \( \tilde{L} = \tilde{L}(\tilde{A}) \) then, \( \tilde{L} = \tilde{L}(\tilde{A}^ \prime) \), where \( \tilde{A}^ \prime \) is a DFA-IVFS.

Let \( \tilde{A} = (Q, \Sigma, \tilde{\delta}, S, F_{\tilde{A}}) \) represents a NDFA-IVFS, we construct a DFA-IVFS \( \tilde{A}^ \prime = (Q', \Sigma, \tilde{\delta}', S', F_{\tilde{A}^ \prime}) \) by using the method of standard subset construction and for each \( P \in Q' (P \subseteq Q) \), define \( F_{\tilde{A}^ \prime}(P) = \max\{ m \mid F_{\tilde{A}}(q) = m, q \in P \} \), where \( m \subseteq [0, 1] \) represents the membership value of the strings in the language.

Hence \( \tilde{L} = \tilde{L}(\tilde{A}^ \prime) \).

### 3.5 INTERVAL-VALUED FUZZY REGULAR EXPRESSIONS (IVFRE’s)

Every string in an IVFRL has finite membership value in \( I[0, 1] \). Set of all strings associated with these subintervals \( [0, 1] \) forms a regular language. IVFRL can be described by a modified regular expressions having membership values in \( I[0, 1] \). This modified interval-valued fuzzy regular expression can be used in lexical analysis of IVFRL.

For example \( (a + b)^*[/0.4,0.8] + ab^* (b + a^*)[/0.6,0.9] \) represents a modified interval-valued fuzzy regular expression.

**Definition 3.5.1:** Let \( \Sigma \) be an alphabet and \( M = I[0, 1] \) a finite set of closed subintervals in \( [0, 1] \).
1. Let ‘e’ be a regular expression over $\Sigma$ and $m \in M$. Then, we call
   \[ e = e / m \]
   an interval-valued fuzzy regular expression (IVFRE), where $m$ represents the membership value of ‘e’.

2. Let $\tilde{e}_1$ and $\tilde{e}_2$ be two IVFREs, then the following will hold.
   a. $\phi \in IVFRE$ with membership value [1, 1],
   b. $\varepsilon \in IVFRE$ with membership [1, 1],
   c. $a \in IVFRE$ with membership $m \in I[0, 1], \forall a \in \Sigma$.

3. $\forall \tilde{e}_1, \tilde{e}_2 \in IVFREs, (\tilde{e}_1 + \tilde{e}_2) \in IVFREs, (\tilde{e}_1 \cdot \tilde{e}_2) \in IVFREs$, and $(\tilde{e}_1)^* \in IVFREs$.
   By applying above mentioned steps ((1) and (2)) finite number of times, an interval-valued fuzzy regular expression can be obtained.

**Definition 3.5.2:** Let $\tilde{e}$ be an IVFRE over an alphabet $\Sigma$. Then, the corresponding language i.e., interval-valued fuzzy regular language (IVFRL) $\tilde{L}(\tilde{e})$ is defined to be:
\[
\tilde{L} = \tilde{L}(\tilde{e}) = \{(x, m) \mid x \in \tilde{L}(e)\}.
\]
Here, $\tilde{L}(e)$ represents the language for regular expression ‘e’ and $m \in I[0,1]$ the membership value of the string $x \in \Sigma^*$. If $\tilde{e} = \tilde{e}_1 + \tilde{e}_2$, $\tilde{e} = (\tilde{e}_1) \cdot (\tilde{e}_2)$ or $\tilde{e} = (\tilde{e}_1)^*$ then,
\[
\tilde{L}(\tilde{e}) = \tilde{L}(\tilde{e}_1) \cup \tilde{L}(\tilde{e}_2), \tilde{L}(\tilde{e}) = \tilde{L}(\tilde{e}_1) \cdot \tilde{L}(\tilde{e}_2) \text{ or } \tilde{L}(\tilde{e}) = (\tilde{L}(\tilde{e}_1))^* \text{ respectively.}
\]

**Definition 3.5.3:** An IVFRE $\tilde{e}$ over an alphabet $\Sigma$ is said to be normalized interval-valued fuzzy regular expression if it is of the form $e_1 / m_1 + e_2 / m_2 + \ldots + e_n / m_n$,
where $e_1, e_2, \ldots, e_n$ represents regular expressions over $\Sigma$ and $m_1, m_2, \ldots, m_n$ are the closed subintervals in [0, 1], $n \geq 1$. Note that, if $m = [1, 1]$ then $e / m$ can simply be written as ‘e’.
We assume that ‘*’ and ‘*’ have higher priorities than ‘/’. Thus, certain pairs of parenthesis can be omitted.

**Example 3.4:** The following are all valid IVFREs.

i. $(a + b \cdot c)^* \cdot (c + \phi) / [0.3, 0.5],$
   ii. $(a + b)^* aa(a + b)^*$,
Following is not a valid IVFRE.

\[(a^*b^*[0,5,0.8])/[0.2,0.4] + ca/[0.4,0.5].\]

**Definition 3.5.4:** An IVFRE \( \tilde{e} \) over an alphabet \( \Sigma \) is called *strictly normalized interval-valued fuzzy regular expression*, if it is normalized. i.e.,

\[\tilde{e} = e_1/m_1 + e_2/m_2 + \ldots + e_n/m_n\]

and for any \( m_i \neq m_j \), \( \tilde{L}(e_i) \cap \tilde{L}(e_j) = \emptyset \).

**Example 3.5:**

i. \( ab/[0,3,0.5] + ac/[0,4,0.7] + bc/[0,5,0.9], \)

ii. \( ((bb + bba)^*a)/[0,5,1] + (\{a,b\}^*[ba])/[0,4,0.8], \)

iii. \( a^*b/[0,4,0.9] + (a + b)^*a(a + b)^*b/[0,3,0.7]. \)

In the above example, i), ii) represents the IVFREs which are strictly normalized and, iii) represents an IVFRE which is not strictly normalized.

It is clear from the above definitions that, the families of languages as represented by IVFREs, normalized IVFREs, and strictly normalized IVFREs, respectively, are same as the family of interval-valued fuzzy regular languages.

### 3.6 MYHILL-NERODE THEOREM FOR IVFRL

State reduction or minimization of fuzzy automata has been studied by many authors. All of them have dealt with classical fuzzy automata and reduction has been done using crisp equivalence relations. Myhill-Nerode theorem for finite automaton is a very powerful tool for minimizing the number of states in a finite automaton. It has been extended for fuzzy (final) states automaton [66] accepting fuzzy regular language. Here, we extend it to interval-valued fuzzy (final) states automaton accepting IVFRL. As a result we arrive at minimized interval-valued fuzzy automaton. The output of this automaton will function like the original one helping us to remove some redundant states.
Theorem (Myhill-Nerode theorem for IVFRL) 3.6:

The following three statements are equivalent to one another.

i. An Interval-valued fuzzy regular language $\tilde{L}$ is accepted by some DFA-IVFS.

ii. $\tilde{L}$ is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.

iii. Let the relation $R_{\tilde{L}} \subseteq \Sigma^+ \times \Sigma^+$ be defined by $xR_{\tilde{L}}y \iff \forall z \in \Sigma^+, f_{\tilde{L}}(xz) = f_{\tilde{L}}(yz)$.

Then $R_{\tilde{L}}$ is an equivalence relation of finite index.

Proof: (i) $\Rightarrow$ (ii) Let $\tilde{L}$ be an IVFRL over an alphabet $\Sigma$.

Assume that $\tilde{L}$ is accepted by some DFA-IVFS $\tilde{A} = (Q, \Sigma, \tilde{\delta}, S, F_{\tilde{A}})$.

Let $R_{\tilde{A}}$ be the equivalence relation $xR_{\tilde{A}}y \iff \tilde{\delta}(S, xz) = \tilde{\delta}(S, yz)$ is right invariant since, for any $z$, $\tilde{\delta}(S, xz) = \tilde{\delta}(S, yz)$ if $\tilde{\delta}(S, x) = \tilde{\delta}(S, y)$. Then the index of $R_{\tilde{A}}$ is finite, since the index is at most the number of states in $Q$. Furthermore, $\tilde{L}$ is the union of those equivalence classes having a string ‘$x$’ such that $\tilde{\delta}(S, x)$ is in $F_{\tilde{A}}$ i.e., the equivalence classes corresponding to final states.

(ii) $\Rightarrow$ (iii) We show that any equivalence relation ‘$E$’ satisfying (ii) is a refinement of $R_{\tilde{L}}$; i.e., some equivalence class of $R_{\tilde{L}}$ will be the superset of every equivalence class ‘$E$’. Thus, the index of $R_{\tilde{L}}$ cannot be greater than the index of ‘$E$’ and so is finite. Assume that ‘$xEy$’.

For each $z \in \Sigma^+$, since ‘$E$’ is right invariant, ‘$xzEy$’ and thus $\tilde{L}(xz) = \tilde{L}(yz)$. Hence $xR_{\tilde{A}}y$.

We conclude that each equivalence class of ‘$E$’ is the subset of some equivalence class of $R_{\tilde{L}}$.

(iii) $\Rightarrow$ (i) To show that $R_{\tilde{L}}$ is right invariant, suppose $xR_{\tilde{A}}y$, and let $w \in \Sigma^+$, we must prove that $xwR_{\tilde{A}}yw$; i.e., for any $z$, $\tilde{L}(xwz) = \tilde{L}(yzw)$. But since $xR_{\tilde{L}}y$, by definition of $R_{\tilde{L}}$ we know that for any $v$, $\tilde{L}(xv) = \tilde{L}(yv)$. We consider ‘$v=wz$’ to prove that $R_{\tilde{L}}$ is right invariant.

Now we present the minimized Interval-valued fuzzy automaton by constructing equivalence classes of $R_{\tilde{L}}$: Let $Q'$ be the finite set of equivalence classes of $R_{\tilde{L}}$ and $[x] \in Q'$ containing $x$. 75
Define \( \tilde{\delta}'([x]a) = [xa] \). Since \( R_L \) is right invariant, this definition is consistent. If we had considered ‘y’ instead of ‘x’ from \([x] \), we would have obtained \( \tilde{\delta}'([x]a) = [ya] \). But \( xR_L y \), so \( \tilde{L}(xz) = \tilde{L}(yz) \). In particular, if \( z = ax' \), \( \tilde{L}(xaz') = \tilde{L}(yaz') \), so \( xaR_L ya \) and \([xa] = [ya] \).

Let \( S' = [\epsilon] \) and let \( F_A' = \{[x] | x \in \tilde{L} \} \). The finite automaton \( \tilde{A}' = (Q', \Sigma, \tilde{\delta}', S', F_A') \) accepts \( \tilde{L} \), since \( \tilde{\delta}'(S', x) = [x] \), and thus \( \tilde{L}(\tilde{A}') = \tilde{L}(\tilde{A}) \).

### 3.7 ALGORITHM FOR MINIMIZATION OF DFA-IVFS

Let \( \tilde{A} = (Q, \Sigma, \tilde{\delta}, S, F_A) \) be a DFA-IVFS. Assume that \( Q = \{q_0, q_1, \ldots, q_n\} \), \( n \geq 0 \) and let \( P = \{(q_i, q_j) \mid q_i, q_j \in Q \text{ and } 0 \leq i < j \leq n\} \).

**begin**

**Step 1:** for each pair \((q_i, q_j) \) \( \in P \), and \( F_A(q_i) \neq F_A(q_j) \) do mark \((q_i, q_j) \);

**Step 2:** for each unmarked pair \((q_i, q_j) \) \( \in P \) do mark

  if for some \( x \in \Sigma \), \( \tilde{\delta}(q_i, x), \tilde{\delta}(q_j, x) \) is marked then

  **Step 2.1:** mark \((q_i, q_j) \);

  **Step 2.2:** recursively mark all unmarked pairs on the list of \((q_i, q_j) \)

  and on the list of other pairs that are marked at this step.

else

  **Step 2.3:** for all input symbols ‘x’ do

  put \((q_i, q_j) \) on the list for \( (\tilde{\delta}(q_i, x), \tilde{\delta}(q_j, x)) \)

  unless \( \tilde{\delta}(q_i, x) = \tilde{\delta}(q_j, x) \).

**Step 3:** Equivalence classes of \( Q \) are constructed as follows;

  For \( i = 0 \) to \( n - 1 \) do

  For \( j = i + 1 \) to \( n \) do

    if \((q_i, q_j) \) is unmarked, \( q_j \) is in \([q_i] \), the equivalence class containing \( q_i \).

**Step 4:** Define minimum DFA-IVFS \( \tilde{A}' = (Q', \Sigma, \tilde{\delta}', S', F_A') \) as follows;

\[
Q' = \{ [q_i] \mid q_i \in Q \}, \quad \tilde{\delta}' ([q_i], a) = [\tilde{\delta}(q_i, a)], \quad S' = [S], \quad F_A' ([q_i]) = F_A (q_i).
\]

**end.**
The complexity of the above algorithm is $O(n^2) + 2O(an^2)$, where $n$ is the number of states of given DFA-IVFS and $a$ the element in the assumed set of input alphabet.

Example 3.6:

Let $\tilde{A}=(Q, \tilde{\delta}, S, F_{\tilde{A}})$ be a DFA-IVFS (Figure 3.3(a)), where $Q=\{a, b, c, d, e, f\}$, $\Sigma=[0,1]$, $\tilde{\delta}:Q \times \Sigma \rightarrow Q$ the transition function given as $\tilde{\delta}(a,0)=b, \tilde{\delta}(a,1)=c, \tilde{\delta}(b,0)=a, \tilde{\delta}(b,1)=d$, $\tilde{\delta}(c,0)=e, \tilde{\delta}(c,1)=f, \tilde{\delta}(d,0)=e, \tilde{\delta}(d,1)=f, \tilde{\delta}(e,0)=f, \tilde{\delta}(e,1)=f, \tilde{\delta}(f,0)=f, \tilde{\delta}(f,1)=f$, $S=\{a\}$ the interval-valued fuzzy starting state with membership value $F_{\tilde{A}}(a)=[0.3,0.7]$, and $F_{\tilde{A}}:Q \rightarrow M$ where $M$ is the set of all closed subintervals in [0, 1] the membership function of interval-valued fuzzy (final) state set given as $F_{\tilde{A}}(b)=[0.3,0.7], F_{\tilde{A}}(c)=[0.5,0.8], F_{\tilde{A}}(d)=[0.5,0.8], F_{\tilde{A}}(e)=[0.5,0.8]$ and $F_{\tilde{A}}(f)=[0.2,0.3]$.

![Figure 3.3 (a): DFA-IVFS](image)
The IVFRL accepted by DFA-IVFS (Figure 3.3 (a) and its minimized DFA-IVFS (Figure 3.3 (b)) is: \( \tilde{L} = \{0^* / [0.3, 0.7], 0^*10^* / [0.5, 0.8], 0^*10^*1(0 + 1)^* / [0.2, 0.3] \} \).

To reduce interval-valued fuzzy sets into fuzzy sets, we will consider only the lower membership of the states. In this case, DFA-IVFS changes to deterministic finite automaton with fuzzy (final) states (DFA-FS). Further, we can reduce this DFA-FS depending on the state transition and membership value of its each state. For reducing DFA-FS we apply the algorithm given in [66]. Again, if fuzzy set is reduced to crisp set, we treat each state with membership value zero in DFA-FS as non-final state and all other states as final states in deterministic finite automaton (DFA). Thus DFA-FS becomes DFA. Again this DFA may get reduced further depending on its states transition and for doing so, we consider the algorithm given in [5].

Moreover, if interval-valued fuzzy set is reduced to fuzzy set, interval-valued fuzzy regular language becomes fuzzy regular language, where each string is having membership value in unit interval. This language can be recognized by DFA-FS & NDFA-FS [66]. Again, on reduction of fuzzy sets into crisp sets, fuzzy regular language becomes regular language and is recognized by DFA & NDFA. Here we will consider only strings of fuzzy regular language having non zero membership as strings of regular language.

The DFA-IVFS (Figure 3.3 (a)) is changed to DFA-FS (Figure 3.4 (a)). The DFA-FS and its minimized one (Figure 3.4 (b)) recognizing the fuzzy regular language \( \tilde{L} = \{0^* / 0.3, 0^*10^* / 0.5, 0^*10^*1(0 + 1)^* / 0.2 \} \) are given by;
Above DFA-FS (Figure 3.4 (a)) is changed to DFA (Figure 3.5 (a)). This DFA and its minimized DFA (Figure 3.5 (b)) will accept the regular language 

\[ \bar{L} = \{ 1^*, 1^0 0, 1^0 1^+, 1^* 01^+ 0 (0 + 1)^*, 1^* 00 (0 + 1)^* \} \]

and are shown in Figures 3.5 (a) & 3.5 (b) respectively.
3.8 CONCLUSION

Interval-valued fuzzy sets are one of the important extensions of fuzzy sets that express the membership of an element to a subinterval of [0, 1]. This chapter explores this idea to the strings of fuzzy language resulting in a new language, Interval-valued fuzzy language. It
generalizes the fuzzy language and provides us with a flexible mathematical framework to cope with imprecise information. Finite automata with interval-valued fuzzy transitions and interval-valued fuzzy (final) states are proposed to study this language. It is described that the proposed language can be expressed through interval-valued fuzzy regular expressions. Minimization of Interval-valued fuzzy automaton is achieved through the extended Myhill-Nerode theorem and an algorithm is given for the same. Thus the minimized interval-valued fuzzy automaton accepts the same language as the original. This language can be applied in the study of lexical analysis, pattern recognition, learning systems and so on.