Chapter 3

Product Cordial Labeling of Graphs
3.1 Introduction

A brief account of cordial labeling is given in Chapter 2. In cordial labeling the edge labels were induced by absolute difference of vertex labels while in product cordial labeling the edge labels are induced by product of vertex labels.

3.2 Product Cordial Labeling of Graphs

The concept of product cordial labeling was introduced by Sundaram et al. [53] and defined as follows.

**Definition 3.2.1.** For a graph $G = (V(G), E(G))$, a vertex labeling function $f : V(G) \rightarrow \{0, 1\}$ induces an edge labeling function $f^* : E(G) \rightarrow \{0, 1\}$ defined as $f^*(e = uv) = f(u)f(v)$. Then $f$ is called a product cordial labeling of graph $G$ if $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. A graph $G$ is called product cordial graph if it admits a product cordial labeling.

**Illustration 3.2.2.** The cycle $C_7$ and its product cordial labeling is shown in Figure 3.1.

![Figure 3.1: The cycle $C_7$ and its product cordial labeling](image-url)
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3.3 Some Known Results on Product Cordial Labeling

Sundaram et al. in [53, 55] have proved that:

- Trees, unicyclic graphs of odd order, triangular snakes, dragons, helms are product cordial graphs.

- $P_m \cup P_n, C_m \cup P_n, P_m \cup K_{1,n}, W_m \cup f_n, K_{1,m} \cup K_{1,n}, W_m \cup K_{1,n}, W_m \cup P_n$ and $W_m \cup C_n$ are product cordial graphs.

- $T(P_n), C_n$ if and only if $n$ is odd; $C_n^{(t)}$ provided $t$ is even or both $t$ and $n$ are even and $P_n^2$ if and only if $n$ is odd are product cordial graphs.

- $K_{m,n}(m,n > 2), P_m \square P_n(m,n > 2)$ and $W_n$ are not product cordial graphs.

- If a $(p,q)$ - graph is product cordial then $q \leq \frac{(p-1)(p+1)}{4} + 1$.

Seoud and Helmi [43] have proved following results on product cordial labeling:

- $K_n$ is not product cordial graph for all $n > 4$.

- The corona of a triangular snake with at least two triangles admits product cordial labeling.

- The $C_4$ - snake is product cordial if and only if the number of 4-cycles is odd.

- $C_m \odot K_n$ is product cordial.

Vaidya and Dani [61] have proved that:

- Graph $< S^{(1)}_n : S^{(2)}_n >$ is product cordial.

- Graph $< S^{(1)}_n : S^{(2)}_n : \ldots : S^{(k)}_n >$ is product cordial except $k$ odd and $n$ even.

- Graph $< K^{(1)}_{1,n} : K^{(2)}_{1,n} >$ is product cordial.
• Graph $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$ is product cordial.

• Graph $< W_n^{(1)} : W_n^{(2)} >$ is product cordial.

• Graph $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ is product cordial except $k$ and $n$ both odd.

The following graphs are proved to be product cordial by Vaidya and Kanani [62, 64]:

• The path union of $k$ copies of $C_n$ except when $k$ is odd and $n$ is even.

• The graph obtained by joining two copies of a cycle by path.

• The path union of an odd number copies of the shadow of $C_n$.

• The graph obtained by joining two copies of the shadow of $C_n$ by a path of arbitrary length.

• The path union of an even number of copies of $C_n(C_n)$.

• The graph obtained by joining two copies of $C_n(C_n)$ by a path of arbitrary length.

• The path union of any number of copies of the Petersen graph.

• The graph obtained by joining two copies of the Petersen graph by a path of arbitrary length.

Vaidya and Vyas [66, 68, 69] have proved that:

• $A(T_n)$ except $n \equiv 3(\text{mod} 4)$, $A(QS_n)$ except $n \equiv 2(\text{mod} 4)$, $DA(T_n)$ and $DA(QS_n)$ are product cordial graphs.

• $S'(B_{n,n})$, duplicating each edge by a vertex in $B_{n,n}$ and duplicating each vertex by an edge in $B_{n,n}$ admits product cordial labeling.

• $D_2(B_{n,n})$ is not product cordial graph.

• $P_m \times P_n$, $C_{2m} \times P_{2n}$ and $C_{2m} \times C_{2n}$ are product cordial graphs.
3.4 Product Cordial Labeling of Some Cycle Related Graphs

**Theorem 3.4.1.** $F_n$ is product cordial.

**Proof.** Let $v'$ be the common vertex of $n$ cycles, $v_1, v_2, \ldots, v_{2n}$ be the other vertices and $e_1, e_2, \ldots, e_{3n}$ be the edges of $F_n$.

Define $f : V(F_n) \to \{0, 1\}$, we consider following two cases.

**Case 1:** When $n$ is even.

\[
\begin{align*}
   f(v_i) &= 0, \quad 1 \leq i \leq n \\
   f(v_i) &= 1, \quad \text{otherwise} \\
   f(v') &= 1.
\end{align*}
\]

In view of the above defined labeling pattern we have,

\[
\begin{align*}
   v_f(0) &= v_f(1) - 1 = n \\
   e_f(0) &= e_f(1) = \frac{3n}{2}
\end{align*}
\]

**Case 2:** When $n$ is odd.

\[
\begin{align*}
   f(v_i) &= 0, \quad 1 \leq i \leq n \\
   f(v_i) &= 1, \quad \text{otherwise} \\
   f(v') &= 1.
\end{align*}
\]

In view of the above defined labeling pattern we have,

\[
\begin{align*}
   v_f(0) + 1 &= v_f(1) = n + 1 \\
   e_f(0) &= e_f(1) + 1 = \left\lceil \frac{3n}{2} \right\rceil
\end{align*}
\]
Thus in each case we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\). Hence \(F_n\) is product cordial.

**Illustration 3.4.2.** The graph \(F_5\) and its product cordial labeling is shown in Figure 3.2.

![Figure 3.2: \(F_5\) and its product cordial labeling](image)

**Theorem 3.4.3.** Cycle \(C_n\) with one chord is product cordial except when \(n\) is even and the chord is joining the vertices which are at diameter distance.

**Proof.** Let \(G\) be the cycle graph with one chord. Let \(v_1, v_2, \ldots, v_n\) be the vertices of \(G\). Here graph \(G\) has \(n\) vertices and \(n + 1\) edges.

Define \(f : V(G) \rightarrow \{0, 1\}\), we consider following two cases.

**Case 1:** When \(n\) is odd.

Without loss of generality we assume that the chord is between vertex \((v_1, v_i)\) where \(3 \leq i \leq \left\lceil \frac{n}{2} \right\rceil\).

\[
\begin{align*}
    f(v_i) &= 1, & 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\
    f(v_i) &= 0, & \text{otherwise.}
\end{align*}
\]

In view of the above defined labeling pattern we have,
\[ v_f(0) + 1 = v_f(1) = \left\lceil \frac{n}{2} \right\rceil \]
\[ e_f(0) = e_f(1) = \frac{n + 1}{2} \]

**Case 2:** When \( n \) is even.

Without loss of generality we assume that the chord is between vertex \((v_1, v_i)\) where \(3 \leq i \leq \frac{n}{2} + 1\).

**Subcase 1:** When the chord is between \((v_1, v_i)\) where \(i = \frac{n}{2} + 1\).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(\frac{n}{2}\) vertices out of total \(n\) vertices. The vertex with label 0 will give rise to at least \(\frac{n}{2} + 2\) edges with label 0 and at most \(\frac{n}{2} - 1\) edges with label 1 out of total \(n + 1\) edges of \(G\). Therefore \(|e_f(0) - e_f(1)| = 3\). Thus the edge condition for a graph to be product cordial is violated.

**Subcase 2:** When the chord is between \((v_1, v_i)\) where \(3 \leq i \leq \frac{n}{2}\).

\[ f(v_i) = 1, \quad 1 \leq i \leq \frac{n}{2} \]
\[ f(v_i) = 0, \quad \text{otherwise.} \]

In view of the above defined labeling pattern we have,

\[ v_f(0) = v_f(1) = \frac{n}{2} \]
\[ e_f(0) = e_f(1) - 1 = \frac{n}{2} + 1 \]

Thus in each case we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\). Hence Cycle \(C_n\) with one chord is a product cordial graph except when \(n\) is even and the chord is joining the vertices which are at diameter distance.

**Illustration 3.4.4.** The cycle \(C_7\) with one chord and its product cordial labeling is shown in Figure 3.3.
Theorem 3.4.5. Cycle \( C_n \) with twin chords is product cordial except when \( n \) is even and a chord joining vertices which are at diameter distance.

Proof. Let \( G \) be the cycle graph with twin chords. Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( G \). Here graph \( G \) has \( n \) vertices and \( n + 2 \) edges.

Define \( f : V(G) \rightarrow \{0, 1\} \), we consider following two cases.

Case 1: When \( n \) is odd.

Without loss of generality we assume that let the chords are between vertex \((v_1, v_i)\) and \((v_1, v_{i+1})\) where \( 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \).

\[
\begin{align*}
f(v_i) &= 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
f(v_i) &= 0, \quad \text{otherwise.}
\end{align*}
\]

In view of the above defined labeling pattern we have,

\[
v_f(0) + 1 = v_f(1) = \left\lfloor \frac{n}{2} \right\rfloor
\]

If the chords are between \((v_1, v_i)\) and \((v_1, v_{i+1})\) where \( i = \left\lfloor \frac{n}{2} \right\rfloor \):

Figure 3.3: The cycle \( C_7 \) with one chord and its product cordial labeling
\[ e_f(0) = e_f(1) + 1 = \left\lceil \frac{n}{2} \right\rceil + 1 \]

If the chords are between \((v_1, v_i)\) and \((v_1, v_{i+1})\) where \(3 \leq i < \left\lceil \frac{n}{2} \right\rceil:\)

\[ e_f(0) + 1 = e_f(1) = \left\lceil \frac{n}{2} \right\rceil + 1 \]

**Case 2:** When \(n\) is even.

Without loss of generality we assume that let the chords are between vertex \((v_1, v_i)\) and \((v_1, v_{i+1})\) where \(3 \leq i \leq \frac{n}{2}\.\)

**Subcase 1:** When the chords are between \((v_1, v_i)\) and \((v_1, v_{i+1})\) where \(i = \frac{n}{2}.\)

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(\frac{n}{2}\) vertices out of total \(n\) vertices. The vertex with label 0 will give rise to at least \(\frac{n}{2} + 2\) edges with label 0 and at most \(\frac{n}{2}\) edges with label 1 out of total \(n + 2\) edges of \(G\). Therefore \(|e_f(0) - e_f(1)| = 2\). Thus the edge condition for a graph to be product cordial is violated.

**Subcase 2:** When the chords are between \((v_1, v_i)\) and \((v_1, v_{i+1})\) where \(3 \leq i < \frac{n}{2}.\)

\[ f(v_i) = 1, \quad 1 \leq i < \frac{n}{2} \]  
\[ f(v_i) = 0, \quad \text{otherwise.} \]

In view of the above defined labeling pattern we have,

\[ v_f(0) = v_f(1) = \frac{n}{2} \]  
\[ e_f(0) = e_f(1) = \frac{n}{2} + 1 \]

Thus in each case we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\). Hence Cycle \(C_n\) with twin chords is product cordial except when \(n\) is even and a chord joining vertices at diameter distance.
Illustration 3.4.6. The cycle $C_8$ with twin chords and its product cordial labeling is shown in Figure 3.4.

![Figure 3.4: The cycle $C_8$ with twin chords and its product cordial labeling](image)

Theorem 3.4.7. $ACr_n$ is a product cordial graph.

Proof. Let $v_1, v_2, \ldots, v_n$ are the vertices of cycle $C_n$ and $u_i, w_i$ be the vertices of path $P_i$. To construct armed crown $ACr_n$ join vertex $v_i$ of cycle $C_n$ with vertex $u_i$ with path. Then $|V(ACr_n)| = 3n$ and $|E(ACr_n)| = 3n$.

To define $f : V(G) \rightarrow \{0, 1\}$ we consider following two cases.

Case 1: When $n$ is odd.

Sundaram et al. [53] have proved that unicyclic graph of odd order is product cordial. Thus $ACr_n$ is product cordial for odd $n$.

Case 2: When $n$ is even.

\[
\begin{align*}
  f(v_i) &= 1 \quad \text{for all } i \\
  f(u_i) &= 0 \quad \text{for } 1 \leq i \leq \frac{n}{2} \\
  f(u_i) &= 1 \quad \text{otherwise} \\
  f(w_i) &= 0 \quad \text{for all } i
\end{align*}
\]
In view of the above defined labeling pattern we have

\[ v_f(0) = v_f(1) = \frac{3n}{2} \]
\[ e_f(0) = e_f(1) = \frac{3n}{2} \]

Thus in each case we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\). Hence \(ACr_n\) is a product cordial graph.

**Illustration 3.4.8.** Armed crown \(ACr_5\) and its product cordial labeling is shown in Figure 3.5.

![Figure 3.5: Armed crown \(ACr_5\) and its product cordial labeling](image)

**Theorem 3.4.9.** The graph obtained by mutual edge duplication in cycle \(C_n\) is product cordial graph except for \(n = 3, 4, 5, 6, 7\).

**Proof.** Let \(\{v_1, v_2 \ldots v_n\}\) be the set of vertices of first copy of cycle \(C_n\) and \(\{u_1, u_2 \ldots u_n\}\) be the set of vertices of second copy of cycle \(C_n\). Without loss of generality we duplicate the edges \(v_1v_n\) and \(u_1u_n\). The resultant graph will have \(2n\) vertices and \(2n + 4\) edges.

To define \(f : V \rightarrow \{0, 1\}\) we consider following two cases.

**Case 1:** When \(n = 3, 4, 5, 6, 7\).
In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of total \( 2n \) vertices. The vertex with label 0 will give rise to at least \( e_f(0) \) edges with label 0 and at most \( e_f(1) \) edges with label 1 as shown in column 6 and column 7 of Table 3.1. Thus the edge condition for a graph to be product cordial is violated as shown in column 8 of Table 3.1.

| \( n \) | \( p \) | \( q \) | \( v_f(0) \) | \( v_f(1) \) | \( e_f(0) \) | \( e_f(1) \) | \( |e_f(0) - e_f(1)| \) |
|---|---|---|---|---|---|---|---|
| 3 | 6 | 10 | 3 | 3 | 7 | 3 | 4 |
| 4 | 8 | 12 | 4 | 4 | 8 | 4 | 4 |
| 5 | 10 | 14 | 5 | 5 | 9 | 5 | 4 |
| 6 | 12 | 16 | 6 | 6 | 10 | 6 | 4 |
| 7 | 14 | 18 | 7 | 7 | 10 | 8 | 2 |

Table 3.1

**Case 2:** When \( n \neq 3, 4, 5, 6, 7 \).

\[
f(v_i) = 0, \quad 3 \leq i \leq n - 2
\]

\[
f(v_i) = 1, \quad \text{otherwise}
\]

\[
f(u_i) = 0, \quad 3 \leq i \leq 6
\]

\[
f(u_i) = 1, \quad \text{otherwise}
\]

In view of the above defined labeling pattern we have,

\[
v_f(0) = v_f(1) = n
\]

\[
e_f(0) = e_f(1) = n + 2
\]

Hence from the case 1 and case 2 we have the required result.

**Illustration 3.4.10.** The graph obtained by mutual edge duplication in cycle \( C_8 \) and its product cordial labeling is shown in Figure 3.6.
Theorem 3.4.11. The graph obtained by mutual vertex duplication in cycle $C_n$ is product cordial graph except for $n = 3, 4, 5$.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of first copy of cycle $C_n$ and $\{u_1, u_2, \ldots, u_n\}$ be the set of vertices of second copy of cycle $C_n$. Without loss of generality we duplicate the vertices $v_1$ and $u_1$. The resultant graph will have $2n$ vertices and $2n + 4$ edges.

To define $f : V \to \{0, 1\}$ we consider following two cases.

**Case 1:** When $n = 3, 4, 5$.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $n$ vertices out of total $2n$ vertices. The vertex with label 0 will give rise to at least $e_f(0)$ edges with label 0 and at most $e_f(1)$ edges with label 1 as shown in column 6 and column 7 of Table 3.2. Thus the edge condition for a graph to be product cordial is violated as shown in column 8 of Table 3.2.

| $n$ | $p$ | $q$ | $v_f(0)$ | $v_f(1)$ | $e_f(0)$ | $e_f(1)$ | $|e_f(0) - e_f(1)|$ |
|-----|-----|-----|----------|----------|----------|----------|-------------------|
| 3   | 6   | 10  | 3        | 3        | 7        | 3        | 3                 |
| 4   | 8   | 12  | 4        | 4        | 8        | 4        | 4                 |
| 5   | 10  | 14  | 5        | 5        | 8        | 6        | 6                 |

**Table 3.2**

**Case 2:** When $n \neq 3, 4, 5$.
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\[
\begin{align*}
  f(u_i) &= 0, \quad 3 \leq i \leq n - 1 \\
  f(u_i) &= 1, \quad \text{otherwise} \\
  f(v_i) &= 0, \quad 3 \leq i \leq 5 \\
  f(v_i) &= 1, \quad \text{otherwise}
\end{align*}
\]

In view of the above defined labeling pattern we have,

\[
\begin{align*}
  v_f(0) &= v_f(1) = n \\
  e_f(0) &= e_f(1) = n + 2
\end{align*}
\]

Hence form the case 1 and case 2 we have the required result.

**Illustration 3.4.12.** The graph obtained by mutual vertex duplication in cycle \(C_7\) and its product cordial labeling is shown in Figure 3.7.

![Figure 3.7: The graph obtained by mutual vertex duplication in cycle \(C_7\) and its product cordial labeling](image)

**Theorem 3.4.13.** The graph obtained by duplication of an arbitrary edge \(e_k\) in cycle \(C_n\) is product cordial graph except for \(n = 4, 5, 6, 7, 8\).

**Proof.** Let \(e_1, e_2, \ldots, e_n\) be the edges of cycle \(C_n\). Without loss of generality we duplicate the edge \(e_1\) by new edge \(e'_1 = v'_1v'_2\). Now the resultant graph will have \(n + 2\) vertices and \(n + 3\) edges.

To define \(f : V \rightarrow \{0, 1\}\) we consider following four cases.

**Case 1:** When \(n = 4, 5, 6, 7, 8\).
In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( \left\lfloor \frac{n+2}{2} \right\rfloor \) vertices out of total \( n + 2 \) vertices. The vertex with label 0 will give rise to at least \( e_f(0) \) edges with label 0 and at most \( e_f(1) \) edges with label 1 as shown in column 6 and column 7 of Table 3.3. Thus the edge condition for a graph to be product cordial is violated as shown in column 8 of Table 3.3.

| \( n \) | \( p \) | \( q \) | \( v_f(0) \) | \( v_f(1) \) | \( e_f(0) \) | \( e_f(1) \) | \( |e_f(0) - e_f(1)| \) |
|---|---|---|---|---|---|---|---|
| 4 | 6 | 7 | 3 | 3 | 5 | 2 | 3 |
| 5 | 7 | 8 | 3 | 4 | 5 | 3 | 2 |
| 6 | 8 | 9 | 4 | 4 | 6 | 3 | 3 |
| 7 | 9 | 10 | 4 | 5 | 6 | 4 | 2 |
| 8 | 10 | 11 | 5 | 5 | 7 | 4 | 3 |

Table 3.3

Case 2: When \( n \neq 4, 5, 6, 7, 8 \).

Sub case 1: When \( n = 3 \) the graph and its product cordial labeling is show in Figure 3.8.

![Figure 3.8: The graph obtained by duplication of an edge in \( C_3 \) and its product cordial labeling](image)

Sub case 2: When \( n \) is odd.

\[
f(v_i) = 0, \quad 4 \leq i \leq \left\lfloor \frac{n+2}{2} \right\rfloor + 3
\]

\[
f(v_i) = 1, \quad \text{otherwise}
\]

\[
f(v'_i) = 1, \quad i = 1, 2
\]

In view of the above defined labeling pattern we have,
\[ v_f(0) + 1 = v_f(1) = \left\lceil \frac{n+2}{2} \right\rceil \]
\[ e_f(0) = e_f(1) = \frac{n+3}{2} \]

**Sub case 3:** When \( n \) is even.

\[ f(v) = 0, \quad 4 \leq i \leq \frac{n+8}{2} \]
\[ f(v_i) = 1, \quad \text{otherwise} \]
\[ f(v_i') = 1, \quad i = 1, 2 \]

In view of the above defined labeling pattern we have,

\[ v_f(0) = v_f(1) = \frac{n+2}{2} \]
\[ e_f(0) - 1 = e_f(1) = \frac{n+2}{2} \]

Hence from the case 1 and case 2 we have the required result.

**Illustration 3.4.14.** The graph obtained by duplication of an edge in \( C_9 \) and its product cordial labeling is shown in Figure 3.9.

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**Figure 3.9:** The graph obtained by duplication of an edge in \( C_9 \) and its product cordial labeling
Theorem 3.4.15. The graph obtained by duplication of an arbitrary vertex by a new edge in cycle \( C_n \) is product cordial.

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be vertices and \( e_1, e_2, \ldots, e_n \) be edges of cycle \( C_n \). Without loss of generality we duplicate the vertex \( v_n \) by an edge \( e' = v'_1 v'_2 \). Let the graph so obtained is \( G \). Then \( |V(G)| = n + 2 \) and \( |E(G)| = n + 3 \).

To define \( f : V(G) \to \{0, 1\} \) we consider following two cases.

**Case 1:** When \( n \) is odd.

\[
f(v_i) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
f(v_i) = 1, \quad \text{otherwise} \\
f(v'_i) = 1, \quad i = 1, 2
\]

In view of the above defined labeling pattern we have

\[
v_f(0) = v_f(1) - 1 = \left\lfloor \frac{n}{2} \right\rfloor \\
e_f(0) = e_f(1) = \frac{n + 3}{2}
\]

**Case 2:** When \( n \) is even.

\[
f(v_i) = 0, \quad 1 \leq i \leq \frac{n}{2} + 1 \\
f(v_i) = 1, \quad \text{otherwise} \\
f(v'_i) = 1, \quad i = 1, 2
\]

In view of the above defined labeling pattern we have

\[
v_f(0) = v_f(1) = \frac{n}{2} + 1 \\
e_f(0) - 1 = e_f(1) = \frac{n}{2} + 1
\]

Thus in each case we have \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

Hence the graph obtained by duplication of an arbitrary vertex by a new edge in cycle \( C_n \) is product cordial. \( \blacksquare \)
Illustration 3.4.16. The graph obtained by duplication of a vertex by an edge in $C_7$ and its product cordial labeling is shown in Figure 3.10.

*Figure 3.10: The graph obtained by duplication of a vertex by an edge in $C_7$ and its product cordial labeling*

Theorem 3.4.17. The graph obtained by duplicating all the vertices by edges in cycle $C_n$ is not product cordial except for $n = 3$.

*Proof.* Let $v_1, v_2, \ldots, v_n$ be vertices and $e_1, e_2, \ldots, e_n$ be edges of cycle $C_n$. Let the graph obtained by duplicating all the vertices by edges in cycle $C_n$ is $G$. Then $|V(G)| = 3n$ and $|E(G)| = 4n$. To prove the result we consider following three cases.

**Case 1:** When $n = 3$.

The graph obtained by duplicating all the vertices by edges in cycle $C_3$ and its product cordial labeling is shown in Figure 3.11.

**Case 2:** When $n$ is odd ($n \neq 3$).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $\left\lfloor \frac{3n}{2} \right\rfloor$ vertices out of $3n$ vertices. The vertices with label 0 will give rise to at least $2n + 1$ edges with label 0 and at most $2n - 1$ edge with label 1 out of total $4n$ edges. Therefore $|e_f(0) - e_f(1)| = 2$. Thus the edge condition for a graph to be product cordial is violated.
Case 3: When $n$ is even.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $\frac{3n}{2}$ vertices out of $3n$ vertices. The vertices with label 0 will give rise to at least $2n + 1$ edges with label 0 and at most $2n - 1$ edge with label 1 out of total $4n$ edges. Therefore $|e_f(0) - e_f(1)| = 2$. Thus the edge condition for a graph to be product cordial is violated.

Hence the graph obtained by duplicating all the vertices by edges in cycle $C_n$ is not product cordial except for $n = 3$. ■

Theorem 3.4.18. The graph obtained by duplication of an arbitrary edge by a new vertex in cycle $C_n$ is product cordial except for $n = 3$.

Proof. Let $v_1, v_2, \ldots, v_n$ be vertices and $e_1, e_2, \ldots, e_n$ be edges of cycle $C_n$. Without loss of generality we duplicate the edge $v_{n-1}v_n$ by a vertex $v'$. Let the graph so obtained is $G$. Then $|V(G)| = n + 1$ and $|E(G)| = n + 2$.

To define $f : V(G) \to \{0, 1\}$ we consider following three cases.

Case 1: When $n$ is odd ($n \neq 3$).

\[
\begin{align*}
f(v_i) &= 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
f(v_i) &= 1, \quad \text{otherwise} \\
f(v') &= 1
\end{align*}
\]
In view of the above defined labeling pattern we have

\[ v_f(0) = v_f(1) = \frac{n + 1}{2} \]
\[ e_f(0) - 1 = e_f(1) = \frac{n + 1}{2} \]

**Case 2:** When \( n \) is even.

\[ f(v_i) = 0, \quad 1 \leq i \leq \frac{n}{2} \]
\[ f(v_i) = 1, \quad \text{otherwise} \]
\[ f(v') = 1 \]

In view of the above defined labeling pattern we have

\[ v_f(0) = v_f(1) - 1 = \frac{n}{2} \]
\[ e_f(0) = e_f(1) = \frac{n}{2} + 1 \]

Thus in case 1 and case 2 we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\).

**Case 3:** When \( n = 3 \).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to two vertices out of four vertices. The vertices with label 0 will give rise to at least four edges with label 0 and at most one edge with label 1 out of total five edges. Therefore \(|e_f(0) - e_f(1)| = 3\). Thus the edge condition for a graph to be product cordial is violated.

Hence the graph obtained by duplication of an arbitrary edge by a new vertex in cycle \( C_n \) is product cordial except for \( n = 3 \).

**Illustration 3.4.19.** The graph obtained by duplication of an edge by a vertex in \( C_8 \) and its product cordial labeling is shown in **Figure 3.12**.

**Theorem 3.4.20.** The graph obtained by duplicating all the edges by vertices in cycle \( C_n \) is not product cordial.
Chapter 3. *Product Cordial Labeling of Graphs*

\[ \text{Figure 3.12: The graph obtained by duplication of an edge by a vertex in } C_8 \text{ and its product cordial labeling} \]

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be vertices and \( e_1, e_2, \ldots, e_n \) be edges of cycle \( C_n \). Let the graph obtained by duplicating all the edge by vertices in cycle \( C_n \) is \( G \). Then \( |V(G)| = 2n \) and \( |E(G)| = 3n \). To prove the result we consider following two cases.

**Case 1:** When \( n \) is odd.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of \( 2n \) vertices. The vertices with label 0 will give rise to at least \( \left\lfloor \frac{3n}{2} \right\rfloor + 1 \) edges with label 0 and at most \( \left\lfloor \frac{3n}{2} \right\rfloor - 1 \) edge with label 1 out of total \( 3n \) edges. Therefore \( |e_f(0) - e_f(1)| = 3 \). Thus the edge condition for a graph to be product cordial is violated.

**Case 2:** When \( n \) is even.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of \( 2n \) vertices. The vertices with label 0 will give rise to at least \( \frac{3n}{2} + 2 \) edges with label 0 and at most \( \frac{3n}{2} - 2 \) edge with label 1 out of total \( 4n \) edges. Therefore \( |e_f(0) - e_f(1)| = 4 \). Thus the edge condition for a graph to be product cordial is violated.

Hence the graph obtained by duplicating all the edges by vertices in cycle \( C_n \) is not product cordial. \( \blacksquare \)
Theorem 3.4.21. \( D_2(C_n) \) is not a product cordial graph.

Proof. The graph \( D_2(C_n) \) has \( 2n \) vertices and \( 4n \) edges. To prove the result we consider following two cases.

Case 1: When \( n = 3 \).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to 3 vertices out of total 6 vertices. The vertices with label 0 will give rise to at least 6 edges with label 0 and at most 3 edges with label 1 out of total 9 edges. Therefore \( |e_f(0) - e_f(1)| = 6 \). Thus the edge condition for a graph to be product cordial is violated.

Case 2: When \( n \neq 3 \).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of total \( 2n \) vertices. The vertices with label 0 will give rise to at least \( 2n + 4 \) edges with label 0 and at most \( 2n - 4 \) edges with label 1 out of total \( 4n \) edges. Therefore \( |e_f(0) - e_f(1)| = 8 \). Thus the edge condition for a graph to be product cordial is violated.

Hence \( D_2(C_n) \) is not a product cordial graph.

Theorem 3.4.22. \( C_2^2 \) is not a product cordial graph.

Proof. The graph \( C_2^2 \) has \( n \) vertices and \( 2n \) edges. To prove the result we consider following two cases.

Case 1: When \( n \) is odd.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( \left\lfloor \frac{n}{2} \right\rfloor \) vertices out of total \( n \) vertices. The vertices with label 0 will give rise to at least \( n + 2 \) edges with label 0 and at most \( n - 2 \) edges with label 1 out of total \( 2n \) edges. Therefore \( |e_f(0) - e_f(1)| = 4 \). Thus the edge condition for a graph to be product cordial is violated.
Case 2: When \( n \) is even.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( \frac{n}{2} \) vertices out of total \( n \) vertices. The vertices with label 0 will give rise to at least \( n + 3 \) edges with label 0 and at most \( n - 3 \) edges with label 1 out of total \( 2n \) edges. Therefore \( |e_f(0) - e_f(1)| = 6 \). Thus the edge condition for a graph to be product cordial is violated.

Hence \( C_n^2 \) is not a product cordial graph.

\[ \square \]

Theorem 3.4.23. \( M(C_n) \) is not a product cordial graph.

Proof. The graph \( M(C_n) \) has \( 2n \) vertices and \( 3n \) edges. In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of total \( 2n \) vertices. The vertices with label 0 will give rise to at least \( 2n \) edges with label 0 and at most \( n \) edges with label 1 out of total \( 3n \) edges. Therefore \( |e_f(0) - e_f(1)| = n \). Thus the edge condition for a graph to be product cordial is violated. Hence \( M(C_n) \) is not a product cordial graph.

\[ \square \]

Theorem 3.4.24. \( S'(C_n) \) is not a product cordial graph.

Proof. The \( S'(C_n) \) has \( 2n \) vertices and \( 3n \) edges. To prove the result we consider following three cases.

Case 1: When \( n \equiv 0 \pmod{4} \).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of total \( 2n \) vertices. The vertices with label 0 will give rise to at least \( \frac{3n}{2} + 2 \) edges with label 0 and at most \( \frac{3n}{2} - 2 \) edges with label 1 out of total \( 3n \) edges. Therefore \( |e_f(0) - e_f(1)| = 4 \). Thus the edge condition for a graph to be product cordial is violated.

Case 2: When \( n \equiv 1 \pmod{4} \) or \( n \equiv 3 \pmod{4} \).
In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of total \( 2n \) vertices. The vertices with label 0 will give rise to at least \( \frac{3n - 1}{2} + 3 \) edges with label 0 and at most \( \frac{3n + 1}{2} - 3 \) edges with label 1 out of total \( 3n \) edges. Therefore \( |e_f(0) - e_f(1)| = 5 \). Thus the edge condition for a graph to be product cordial is violated.

Case 3: When \( n \equiv 2 \pmod{4} \).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of total \( 2n \) vertices. The vertices with label 0 will give rise to at least \( \frac{3n}{2} + 3 \) edges with label 0 and at most \( \frac{3n}{2} - 3 \) edges with label 1 out of total \( 3n \) edges. Therefore \( |e_f(0) - e_f(1)| = 6 \). Thus the edge condition for a graph to be product cordial is violated.

Hence \( S'(C_n) \) is not a product cordial graph.

3.5 Product Cordial Labeling of Some Path Related Graphs

Theorem 3.5.1. \( M(P_n) \) is Product Cordial.

Proof. Let \( v_1, v_2, \ldots, v_n \) be the vertices of path \( P_n \) and \( v'_1, v'_2, \ldots, v'_{n-1} \) be the vertices added corresponding to the edges \( e_1, e_2, \ldots, e_{n-1} \) in order to obtain \( M(P_n) \). Then \( M(P_n) \) has \( 2n - 1 \) vertices and \( 3n - 4 \) edges.

Define \( f : V(M(P_n)) \to \{0, 1\} \), we consider following two cases.

Case 1: When \( n \) is even.

\[
\begin{align*}
f(v_i) = 0, & \quad 1 \leq i \leq \frac{n}{2} \\
f(v_i) = 1, & \quad \text{otherwise}
\end{align*}
\]

\[
\begin{align*}
f(v'_i) = 0, & \quad 1 \leq i \leq \frac{n}{2} - 1 \\
f(v'_i) = 1, & \quad \text{otherwise}
\end{align*}
\]
In view of the above defined labeling pattern we have,

\[ v_f(0) + 1 = v_f(1) = n \]
\[ e_f(0) = e_f(1) = \frac{3n - 4}{2} \]

**Case 2:** When \( n \) is odd.

\[ f(v_i) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \]
\[ f(v_i) = 1, \quad \text{otherwise} \]
\[ f(v'_i) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \]
\[ f(v'_i) = 1, \quad \text{otherwise} \]

In view of the above defined labeling pattern we have,

\[ v_f(0) + 1 = v_f(1) = n \]
\[ e_f(0) - 1 = e_f(1) = \left\lfloor \frac{3n - 4}{2} \right\rfloor \]

Thus in each case we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\). Hence \( M(P_n) \) is product cordial.

**Illustration 3.5.2.** The graph \( M(P_7) \) and its product cordial labeling is shown in Figure 3.13.

---

**Figure 3.13:** The graph \( M(P_7) \) and its product cordial labeling
Theorem 3.5.3. $D_2(P_n)$ is not a product cordial graph.

Proof. The graph $D_2(P_n)$ has $2n$ vertices and $4n - 4$ edges. In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $n$ vertices out of total $2n$ vertices. The vertices with label 0 will give rise to at least $2n$ edges with label 0 and at most $2n - 4$ edges with label 1 out of total $4n - 4$ edges. Therefore $|e_f(0) - e_f(1)| = 4$. Thus the edge condition for a graph to be product cordial is violated. Hence $D_2(P_n)$ is not a product cordial graph. ■

Theorem 3.5.4. $S'(P_n)$ is a product cordial graph except for odd $n$.

Proof. Let $v_1, v_2, \ldots, v_n$ be vertices and $e_1, e_2, \ldots, e_{n-1}$ be edges of path $P_n$. To obtain $S'(P_n)$ let added vertices are $v'_1, v'_2, \ldots, v'_n$ corresponding to $v_1, v_2, \ldots, v_n$. Let $G = S'(P_n)$. Then $|V(G)| = 2n$ and $|E(G)| = 3n - 3$.

To define $f : V(G) \to \{0, 1\}$ we consider following three cases.

Case 1: When $n = 2$.

$S'(P_2)$ and its product cordial labeling is shown in Figure 3.14.

![Figure 3.14: $S'(P_2)$ and its product cordial labeling](image)

Case 2: When $n$ is even ($n \neq 2$).

$$f(v_i) = 1, \quad 1 \leq i \leq \frac{n}{2} + 1$$

$$f(v_i) = 0, \quad \text{otherwise}$$

$$f(v'_i) = 1, \quad 2 \leq i \leq \frac{n}{2}$$

$$f(v'_i) = 0, \quad \text{otherwise}$$
In view of the above defined labeling pattern we have

\[
\begin{align*}
    v_f(0) &= v_f(1) = n \\
    e_f(0) &= e_f(1) + 1 = \frac{3n - 2}{2}
\end{align*}
\]

Thus in case 1 and case 2 we have \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

**Case 3:** When \( n \) is odd.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of \( 2n \) vertices. The vertices with label 0 will give rise to at least \( \frac{3n - 1}{2} \) edges with label 0 and at most \( \frac{3n - 5}{2} \) edge with label 1 out of total \( 3n - 3 \) edges. Therefore \( |e_f(0) - e_f(1)| = 2 \). Thus the edge condition for a graph to be product cordial is violated.

Hence we conclude that \( S'(P_n) \) is a product cordial graph except for odd \( n \). \( \blacksquare \)

**Illustration 3.5.5.** \( S'(P_6) \) and its product cordial labeling is shown in Figure 3.15.

![Figure 3.15: \( S'(P_6) \) and its product cordial labeling](image)

**Theorem 3.5.6.** The graph obtained by duplication of an arbitrary edge by a new vertex in path \( P_n \) is product cordial except for \( n = 3 \).

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be vertices and \( e_1, e_2, \ldots, e_{n-1} \) be edges of path \( P_n \). Let the graph obtained by duplication of an edge in \( P_n \) by a vertex \( v' \) is \( G \). Then \( |V(G)| = n + 1 \) and \( |E(G)| = n + 1 \).

To define \( f : V(G) \rightarrow \{0, 1\} \) we consider following four cases.

**Case 1:** When \( n = 2 \).
The graph obtained by duplication of an edge by a vertex in path $P_2$ is same as $C_3$, which is product cordial as proved in [53].

**Case 2:** When $n$ is odd ($n \neq 3$).

Without loss of generality we assume that the duplicating edge would be $e_k$, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

$$f(v_i) = 1, \quad k \leq i \leq k + \left\lfloor \frac{n}{2} \right\rfloor - 1$$

$$f(v_i) = 0, \quad \text{otherwise}$$

$$f(v') = 1$$

In view of the above defined labeling pattern we have

$$v_f(0) = v_f(1) = \frac{n+1}{2}$$

$$e_f(0) = e_f(1) = \frac{n+1}{2}$$

**Case 3:** When $n$ is even ($n \neq 2$).

Without loss of generality we assume that the duplicating edge would be $e_k$, $1 \leq k \leq \frac{n}{2}$.

$$f(v_i) = 1, \quad k \leq i \leq k + \frac{n}{2} - 1$$

$$f(v_i) = 0, \quad \text{otherwise}$$

$$f(v') = 1$$

In view of the above defined labeling pattern we have

$$v_f(0) = v_f(1) - 1 = \frac{n}{2}$$

$$e_f(0) = e_f(1) - 1 = \frac{n}{2}$$

Thus in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

**Case 4:** When $n = 3$. 


In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to 2 vertices out of 4 vertices. The vertices with label 0 will give rise to at least 3 edges with label 0 and at most 1 edge with label 1 out of total 4 edges. Therefore $|e_f(0) - e_f(1)| = 2$. Thus the edge condition for a graph to be product cordial is violated.

Hence the graph obtained by duplication of an arbitrary edge by a new vertex in path $P_n$ is product cordial except for $n = 3$.

**Illustration 3.5.7.** The graph obtained by duplication of an edge by a vertex in $P_6$ and its product cordial labeling is shown in Figure 3.16.

![Figure 3.16: The graph obtained by duplication of an edge by a vertex in $P_6$ and its product cordial labeling](image)

**Theorem 3.5.8.** The graph obtained by duplication of an arbitrary vertex by a new edge in path $P_n$ is product cordial except for $n = 2$.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be vertices and $e_1, e_2, \ldots, e_{n-1}$ be edges of path $P_n$. Let the graph obtained by duplication of an arbitrary vertex by a new edge $e' = v'_1v'_2$ is $G$. Then $|V(G)| = n + 2$ and $|E(G)| = n + 2$.

To define $f : V(G) \to \{0, 1\}$ we consider following three cases.

**Case 1:** When $n$ is odd.

**Subcase 1:** When vertex $v_k$, $k = \left\lceil \frac{n}{2} \right\rceil$ is duplicated.

$$f(v_i) = 1, \quad \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq 2 \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(v) = 0, \quad \text{otherwise}$$

$$f(v'_i) = 1, \quad i = 1, 2$$
In view of the above defined labeling pattern we have

\[ v_f(0) = v_f(1) - 1 = \left\lfloor \frac{n}{2} \right\rfloor \]
\[ e_f(0) = e_f(1) - 1 = \left\lfloor \frac{n}{2} \right\rfloor \]

**Subcase 2:** When vertex \( v_k, k \neq \left\lfloor \frac{n}{2} \right\rfloor \) is duplicated.

Without loss of generality we assume that the duplicating vertex would be \( v_k, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \).

\[ f(v_i) = 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \]
\[ f(v_i) = 0, \quad \text{otherwise} \]
\[ f(v'_i) = 1, \quad i = 1, 2 \]

In view of the above defined labeling pattern we have

\[ v_f(0) = v_f(1) - 1 = \left\lfloor \frac{n}{2} \right\rfloor \]
\[ e_f(0) = e_f(1) - 1 = \left\lfloor \frac{n}{2} \right\rfloor \]

**Case 2:** When \( n \) is even \((n \neq 2)\).

Without loss of generality we assume that the duplicating vertex would be \( v_k, 1 \leq k \leq \frac{n}{2} \).

\[ f(v_i) = 1, \quad k \leq i \leq k + \frac{n}{2} - 2 \]
\[ f(v_i) = 0, \quad \text{otherwise} \]
\[ f(v'_i) = 1, \quad i = 1, 2 \]

In view of the above defined labeling pattern we have

\[ v_f(0) = v_f(1) = \frac{n}{2} + 1 \]
\[ e_f(0) = e_f(1) = \frac{n}{2} + 1 \]
Thus in case 1 and case 2 we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

**Case 3:** When $n = 2$.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to 2 vertices out of 4 vertices. The vertices with label 0 will give rise to at least 3 edges with label 0 and at most 1 edge with label 1 out of total 4 edges. Therefore $|e_f(0) - e_f(1)| = 2$. Thus the edge condition for a graph to be product cordial is violated.

Hence the graph obtained by duplication of an arbitrary vertex by a new edge in path $P_n$ is product cordial except for $n = 2$. ■

**Illustration 3.5.9.** The graph obtained by duplication of a vertex by an edge in $P_7$ and its product cordial labeling is shown in Figure 3.17.

![Figure 3.17: The graph obtained by duplication of a vertex by an edge in $P_7$ and its product cordial labeling](image)

**Theorem 3.5.10.** The graph obtained by duplicating all the vertices by edges in path $P_n$ is product cordial.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be vertices and $e_1, e_2, \ldots, e_{n-1}$ be edges of path $P_n$. Let the graph obtained by duplicating all the vertices by edges in path $P_n$ is $G$. Then $|V(G)| = 3n$ and $|E(G)| = 4n - 1$. Let the edge so added corresponding to vertex $v_n$ has end vertices as $v'_n$ and $v''_n$.

To define $f : V(G) \to \{0, 1\}$ we consider following two cases.

**Case 1:** When $n$ is odd.
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$f(v_i) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$f(v_i) = 1, \quad \text{otherwise}$

$f(v'_i) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$f(v'_i) = 1, \quad \text{otherwise}$

$f(v''_i) = 0, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$f(v''_i) = 1, \quad \text{otherwise}$

In view of the above defined labeling pattern we have

$v_f(0) = v_f(1) - 1 = \left\lfloor \frac{3n}{2} \right\rfloor$

$e_f(0) = e_f(1) + 1 = 2n$

Case 2: When $n$ is even.

$f(v_i) = 0, \quad 1 \leq i \leq \frac{n}{2}$

$f(v_i) = 1, \quad \text{otherwise}$

$f(v'_i) = 0, \quad 1 \leq i \leq \frac{n}{2}$

$f(v'_i) = 1, \quad \text{otherwise}$

$f(v''_i) = 0, \quad 1 \leq i \leq \frac{n}{2}$

$f(v''_i) = 1, \quad \text{otherwise}$

In view of the above defined labeling pattern we have

$v_f(0) = v_f(1) = \frac{3n}{2}$

$e_f(0) = e_f(1) + 1 = 2n$

Thus in case 1 and case 2 we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence the graph obtained by duplicating all the vertices by edges in path $P_n$ is product cordial.

Illustration 3.5.11. The graph obtained by duplicating all the vertices by edges in path $P_5$ and its product cordial labeling is shown in Figure 3.18.
**Figure 3.18:** The graph obtained by duplicating all the vertices by edges in path $P_3$ and its product cordial labeling.

**Theorem 3.5.12.** $S_n$ is a product cordial graph for odd $n$ and not product cordial for even $n$.

**Proof.** Let $v_1, v_2, \ldots, v_n$ are the vertices of shell $S_n$ with $v_1$ as a apex vertex. Then $|V(S_n)| = n$ and $|E(S_n)| = 2n - 3$.

To define $f : V(S_n) \rightarrow \{0, 1\}$ we consider following two cases.

**Case 1:** When $n$ is odd.

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 0 & \text{otherwise} \end{cases}$$

In view of the above defined labeling pattern we have

$$1 + v_f(0) = v_f(1) = \left\lfloor \frac{n}{2} \right\rfloor$$

$$e_f(0) = e_f(1) + 1 = \left\lfloor \frac{2n - 3}{2} \right\rfloor$$

Thus in case 1 we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Therefore $S_n$ is a product cordial graph for odd $n$.

**Case 2:** When $n$ is even.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $\frac{n}{2}$ vertices out of $n$ vertices. The vertices with label 0 will give rise
to at least $n$ edges with label 0 and at most $n - 3$ edge with label 1 out of total $2n - 3$ edges. Therefore $|e_f(0) - e_f(1)| = 3$. Thus the edge condition for a graph to be product cordial is violated. Therefore $S_n$ is not a product cordial graph for even $n$.

Hence $S_n$ is a product cordial graph for odd $n$ and not product cordial for even $n$. ■

**Illustration 3.5.13.** Shell $S_7$ and its product cordial labeling is shown in FIGURE 3.19.

![Figure 3.19: Shell $S_7$ and its product cordial labeling](image)

**Theorem 3.5.14.** Double triangular snake $DT_n$ is a product cordial graph for odd $n$ and not a product cordial graph for even $n$.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of path $P_n$ and $v_{n+1}, v_{n+2}, \ldots, v_{3n-2}$ be the newly added vertices in order to obtain $DT_n$. Then $|V(DT_n)| = 3n - 2$ and $|E(DT_n)| = 5n - 5$.

To define $f : V(DT_n) \to \{0, 1\}$ we consider following two cases.

**Case 1:** When $n$ is odd.

- $f(v_i) = 0$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$;
- $f(v_i) = 0$, for $n + 1 \leq i \leq n + \left\lfloor \frac{n}{2} \right\rfloor$;
- $f(v_i) = 1$, otherwise.

In view of the above defined labeling pattern we have
\[ v_f(0) + 1 = v_f(1) = \left\lceil \frac{3n - 2}{2} \right\rceil \]

\[ e_f(0) - 1 = e_f(1) = \frac{5n - 5}{2} \]

Thus we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\).

**Case 2:** When \(n\) is even.

**Subcase 1:** When \(n = 2\).

The graph \(DT_2\) has \(p = 4\) vertices and \(q = 5\) edges. Since \(\frac{p^2 - 1}{4} + 1 = \frac{19}{4} < q\). Thus the necessary condition for a graph to be product cordial is violated. Hence \(DT_2\) is not a product cordial graph.

**Subcase 2:** When \(n \neq 2\).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(\frac{3n - 2}{2}\) vertices out of \(3n - 2\) vertices. The vertices with label 0 will give rise to at least \(\frac{5n}{2} - 1\) edges with label 0 and at most \(\frac{5n}{2} - 4\) edge with label 1 out of total \(5n - 5\) edges. Therefore \(|e_f(0) - e_f(1)| = 3\). Thus the edge condition for a graph to be product cordial is violated.

Hence Double triangular snake \(DT_n\) is a product cordial graph for odd \(n\) and not a product cordial graph for even \(n\).

**Illustration 3.5.15.** The double triangular snake \(DT_7\) and its product cordial labeling is shown in FIGURE 3.20.

![Figure 3.20: The double triangular snake \(DT_7\) and its product cordial labeling](image-url)
3.6 Product Cordial Labeling of Line Graph of Some Graphs

We recall the Definition 2.2.35 of line graph of a graph $G$. We investigate product cordial labeling for line graph of some graphs.

**Theorem 3.6.1.** The graph $L(M(P_n))$ is product cordial for odd $n$ and not product cordial for even $n$.

**Proof.** Let $e_1, e_2, \ldots, e_{n-2}$ be the edges of path $P_{n-1}$ in $M(P_n)$ and $e'_1, e'_2, \ldots, e'_{2n-2}$ be the added edges for the construction of $M(P_n)$. Then $V(L(M(P_n))) = \{e_1, e_2, \ldots, e_{n-2}, e'_1, e'_2, \ldots, e'_{2n-2}\}$. Hence $|V(L(M(P_n)))| = 3n - 4$ and $|E(L(M(P_n)))| = 7n - 14$.

To define $f : V(L(M(P_n))) \rightarrow \{0, 1\}$, we consider following two cases.

**Case 1:** When $n$ is odd.

\[
\begin{align*}
    f(e_i) &= 1; \quad 1 \leq i \leq \frac{n-1}{2}, \\
    f(e_i) &= 0; \quad \frac{n+1}{2} \leq i \leq n-2, \\
    f(e'_i) &= 1; \quad 3 \leq i \leq n+1, \\
    f(e'_i) &= 0; \quad n+2 \leq i \leq 2(n-1) \text{ and } i = 1,2.
\end{align*}
\]

In view of the above defined labeling pattern we have,

\[
\begin{align*}
    v_f(1) &= v_f(0) + 1 = \frac{3n - 3}{2} \\
    e_f(1) + 1 &= e_f(0) = \frac{7n - 13}{2}
\end{align*}
\]

Thus we have $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$.

**Case 2:** When $n$ is even.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $\frac{3n-4}{2}$ vertices out of $3n-4$ vertices. The vertices with label 0 will
give rise to at least \( \frac{7n-12}{2} \) edges with label 0 and at most \( \frac{7n-16}{2} \) edges with label 1 out of total \( 7n-14 \) edges. Therefore \( |e_f(1) - e_f(0)| \geq 2 \). Thus the edge condition for a graph to be product cordial is violated.

Hence the graph \( L(M(P_n)) \) is product cordial for odd \( n \) and not product cordial for even \( n \).

\[\boxed{\text{Illustration 3.6.2.} \text{ The line graph of } M(P_5) \text{ and its product cordial labeling is shown in Figure 3.21.}}\]

\[\text{Figure 3.21: The graph } L(M(P_3)) \text{ and its product cordial labeling}\]

\[\textbf{Theorem 3.6.3.} \text{ The graph } L(T_n) \text{ is product cordial for even } n \text{ and not product cordial for odd } n.\]

\[\text{Proof.} \text{ Let } e_1,e_2,\ldots,e_{n-1} \text{ be the edges of } T_n \text{ corresponding to path } P_n \text{ and } e'_1,e'_2,\ldots, e'_{2n-2} \text{ be the added edges to path } P_n \text{ for the construction of } T_n. \text{ Then } V(L(T_n)) = \{e_1,e_2,\ldots,e_{n-1},e'_1,e'_2,\ldots,e'_{2n-2}\}. \text{ Hence } |V(L(T_n))| = 3n - 3 \text{ and } |E(L(T_n))| = 7n - 11.\]

To define \( f : V(L(T_n)) \to \{0,1\} \), we consider following two cases.

\[\text{Case 1: When } n \text{ is even.}\]

\[f(e_i) = 1; \quad 1 \leq i \leq \frac{n}{2},\]

\[f(e_i) = 0; \quad \frac{n}{2} + 1 \leq i \leq n - 1,\]

\[f(e'_1) = 0;\]

\[f(e'_i) = 1; \quad 2 \leq i \leq n,\]

\[f(e'_i) = 0; \quad n + 1 \leq i \leq 2(n-1).\]
In view of the above defined labeling pattern we have,

\[ v_f(1) = v_f(0) + 1 = \frac{3n - 2}{2} \]

\[ e_f(1) + 1 = e_f(0) = \frac{7n - 10}{2} \]

Thus we have \(|v_f(1) - v_f(0)| \leq 1\) and \(|e_f(1) - e_f(0)| \leq 1\).

**Case 2:** When \(n\) is odd.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(\frac{3n - 3}{2}\) vertices out of \(3n - 3\) vertices. The vertices with label 0 will give rise to at least \(\frac{7n - 7}{2}\) edges with label 0 and at most \(\frac{7n - 15}{2}\) edges with label 1 out of total \(7n - 11\) edges. Therefore \(|e_f(1) - e_f(0)| \geq 4\). Thus the edge condition for a graph to be product cordial is violated.

Hence the graph \(L(T_n)\) is product cordial for even \(n\) and not product cordial for odd \(n\). \[\blacksquare\]

**Illustration 3.6.4.** The graph \(L(T_4)\) and its product cordial labeling is shown in **Figure 3.22**.

![Figure 3.22: The graph L(T_4) and its product cordial labeling](image)

**Theorem 3.6.5.** The graph \(L(ACr_n)\) is a product cordial graph.

**Proof.** For the graph \(ACr_n\), let \(e_1, e_2, \ldots, e_n\) be the edges of \(C_n\) and \(e'_1, e'_2, \ldots, e'_n\) be the edges corresponding to paths and \(e''_1, e''_2, \ldots, e''_n\) be the edges joining path and cycle. Then \(V(L(ACr_n)) = \{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_n, e''_1, e''_2, \ldots, e''_n}\).

Hence \(|V(L(ACr_n))| = 3n\) and \(|E(L(ACr_n))| = 4n\).
To define $f : V(L(ACr_n)) \rightarrow \{0, 1\}$ we consider following two cases.

**Case 1:** When $n$ is odd.

$$f(e_i) = 1; \quad 1 \leq i \leq n,$$
$$f(e'_1) = 1;$$
$$f(e'_i) = 0; \quad 2 \leq i \leq n,$$
$$f(e''_1) = 1; \quad 1 \leq i \leq \frac{n-1}{2},$$
$$f(e''_i) = 0; \quad \frac{n+1}{2} \leq i \leq n.$$

In view of the above defined labeling pattern we have,

$$v_f(1) = v_f(0) + 1 = \frac{3n+1}{2}$$
$$e_f(1) = e_f(0) = 2n$$

**Case 2:** When $n$ is even.

$$f(e_i) = 1; \quad 1 \leq i \leq n,$$
$$f(e'_i) = 0; \quad 1 \leq i \leq n,$$
$$f(e''_i) = 1; \quad 1 \leq i \leq \frac{n}{2},$$
$$f(e'''_i) = 0; \quad \frac{n}{2} + 1 \leq i \leq n.$$

In view of the above defined labeling pattern we have,

$$v_f(1) = v_f(0) = \frac{3n}{2}$$
$$e_f(1) = e_f(0) = 2n$$

Thus in both the cases we have $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. Hence the graph $L(ACr_n)$ is a product cordial graph.

**Illustration 3.6.6.** The line graph of $ACr_5$ and its product cordial labeling is shown in Figure 3.23.
Theorem 3.6.7. The graph $L(P_n^2)$ is not product cordial for odd $n > 3$.

Proof. For the graph $L(P_n^2)$ for odd $n > 3$, $|V(L(P_n^2))| = 2n - 3$ and $|E(L(P_n^2))| = 6n - 16$. In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $\left\lfloor\frac{2n - 3}{2}\right\rfloor$ vertices out of $2n - 3$ vertices. The vertices with label 0 will give rise to at least $\frac{7n - 19}{2}$ edges with label 0 and at most $\frac{5n - 13}{2}$ edges with label 1 out of total $6n - 16$ edges. Therefore $|e_f(1) - e_f(0)| \geq n - 3 > 1$. Thus the edge condition for a graph to be product cordial is violated.

Hence the graph $L(P_n^2)$ is not product cordial for odd $n > 3$. ■

Remark 3.6.8. The graph $P_3^2$ is same as $C_3$ and hence $L(P_3^2) = C_3$, which is product cordial as proved in [53].

Theorem 3.6.9. The graph $L(S'(P_n))$ is not product cordial for even $n > 2$.

Proof. For the graph $L(S'(P_n))$ for even $n > 2$, $|V(L(S'(P_n)))| = 3n - 3$ and $|E(L(S'(P_n))))| = 7n - 12$. In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $\left\lfloor\frac{3n - 3}{2}\right\rfloor$ vertices out of $3n - 3$ vertices. The vertices with label 0 will give rise to at least $\frac{7n - 10}{2}$ edges with label 0 and at most
edges with label 1 out of total \(7n - 12\) edges. Therefore \(|e_f(1) - e_f(0)| \geq 2\).

Thus the edge condition for a graph to be product cordial is violated.

Hence the graph \(L(S'(P_n))\) is not product cordial for even \(n > 2\).

**Remark 3.6.10.** The graph \(S'(P_2)\) is same as \(P_4\) and hence \(L(S'(P_2)) = P_3\), which is product cordial as proved in [53].

**Theorem 3.6.11.** The graph \(L(T(P_n))\) is not product cordial graph.

**Proof.** For the graph \(L(T(P_n))\), \(|V(L(T(P_n))))| = 4n - 5\) and \(|E(L(T(P_n)))))| = 12n - 22\).

To prove the result we consider following two cases.

**Case 1:** When \(n\) is odd.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(\frac{4n-5}{2}\) vertices out of \(4n - 5\) vertices. The vertices with label 0 will give rise to at least \(7n - 13\) edges with label 0 and at most \(5n - 9\) edges with label 1 out of total \(12n - 22\) edges. Therefore \(|e_f(1) - e_f(0)| \geq 2n - 4 > 1\). Thus the edge condition for a graph to be product cordial is violated.

**Case 2:** When \(n\) is even.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(\left\lfloor \frac{4n-5}{2} \right\rfloor\) vertices out of \(4n - 5\) vertices. The vertices with label 0 will give rise to at least \(7n - 12\) edges with label 0 and at most \(5n - 10\) edges with label 1 out of total \(12n - 22\) edges. Therefore \(|e_f(1) - e_f(0)| \geq 2n - 2 > 1\). Thus the edge condition for a graph to be product cordial is violated.

Hence the graph \(L(T(P_n))\) is not product cordial graph.
Chapter 3. Product Cordial Labeling of Graphs

3.7 Product Cordial Labeling of Degree Splitting Graph of Some Graphs

We recall the Definition 2.2.56 of degree splitting graph and prove the following results.

Theorem 3.7.1. The graph $DS(P_n)$ is product cordial graph for $n > 2$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$. For the graph $DS(P_n)$ added vertices are $w_1$ and $w_2$ and added edges are $v_1w_1$, $v_nw_1$ and $v_iw_2$ for $i = 2, 3, \ldots n - 1$. $|V(DS(P_n))| = n + 2$ and $|E(DS(P_n))| = 2n - 1$.

To define $f : V(DS(P_n)) \rightarrow \{0, 1\}$, we consider following two cases.

Case 1: When $n$ is even and $n > 2$.

$$f(v_1) = 0;$$
$$f(v_i) = 1; \quad 2 \leq i \leq \frac{n}{2} + 1$$
$$f(v_i) = 0; \quad \frac{n}{2} + 2 \leq i \leq n$$
$$f(w_1) = 0;$$
$$f(w_2) = 1.$$

In view of the above defined labeling pattern we have

$$v_f(0) = v_f(1) = \frac{n}{2} + 1$$
$$e_f(0) = e_f(1) + 1 = n$$

Case 2: When $n$ is odd.

$$f(v_1) = 0;$$
$$f(v_i) = 1; \quad 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1$$
$$f(v_i) = 0; \quad \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n$$
$$f(w_1) = 0;$$
$$f(w_2) = 1.$$
In view of the above defined labeling pattern we have

\[ v_f(0) = v_f(1) - 1 = \frac{n+1}{2} \]
\[ e_f(0) + 1 = e_f(1) = n \]

Thus in both the cases described above \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\). Hence the graph \(DS(P_n)\) is product cordial graph for \(n > 2\).

**Illustration 3.7.2.** The graph \(DS(P_6)\) and its product cordial labeling is shown in Figure 3.24.

![Figure 3.24: The graph DS(P_6) and its product cordial labeling](image)

**Theorem 3.7.3.** The graph \(DS(S_n)\) is product cordial for odd \(n\) and not product cordial for even \(n\).

**Proof.** Let \(v_1, v_2, \ldots, v_n\) be the vertices of \(S_n\). For the graph \(DS(S_n)\) added vertices are \(w_1\) and \(w_2\) and

\[ E(DS(S_n)) = \{v_i v_{i+1} / 1 \leq i \leq n - 1\} \cup \{v_i v_{i+1} / 3 \leq i \leq n - 1\} \cup \{v_i w_i / 3 \leq i \leq n - 1\} \cup \{v_i v_{i+1}, v_i w_1, v_n w_i\}. \]

For \(n \neq 4\) we have \(|V(DS(S_n))| = n + 2\) and \(|E(DS(S_n))| = 3n - 4\).

To define \(f : V(DS(P_n)) \rightarrow \{0, 1\}\), we consider following two cases.

**Case 1:** When \(n\) is odd.
\[ f(v_1) = 1; \]
\[ f(v_2) = 0; \]
\[ f(v_i) = 1; \quad 3 \leq i \leq \frac{n + 3}{2} \]
\[ f(v_i) = 0; \quad \frac{n + 5}{2} \leq i \leq n \]
\[ f(w_1) = 0; \]
\[ f(w_2) = 1. \]

In view of the above defined labeling pattern we have

\[ v_f(0) + 1 = v_f(1) = \frac{n + 3}{2} \]
\[ e_f(0) = e_f(1) + 1 = \frac{3n - 3}{2} \]

Thus we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\).

**Case 2:** When \(n\) is even.

**Subcase 1:** When \(n = 4\).

For the graph \(DS(S_4)\), \(|V(DS(S_4))| = 6\) and \(|E(DS(S_4))| = 9\). In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to 3 vertices out of 6 vertices. The vertices with label 0 will give rise to at least 6 edges with label 0 and at most 3 edges with label 1 out of total 9 edges. Therefore \(|e_f(0) - e_f(1)| = 3\). Thus the edge condition for a graph to be product cordial is violated.

**Subcase 2:** When \(n\) is even and \(n > 4\).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(\frac{n + 2}{2}\) vertices out of \(n + 2\) vertices. The vertices with label 0 will give rise to at least \(\frac{3n}{2}\) edges with label 0 and at most \(\frac{3n}{2} - 4\) edges with label 1 out of total \(3n - 4\) edges. Therefore \(|e_f(0) - e_f(1)| = 4\). Thus the edge condition for a graph to be product cordial is violated.
Hence the graph $DS(S_n)$ is product cordial for odd $n$ and not product cordial for even $n$.

**Illustration 3.7.4.** The graph $DS(S_7)$ and its product cordial labeling is shown in Figure 3.25.

![Graph DS(S_7) and its product cordial labeling](image)

**Figure 3.25:** The graph $DS(S_7)$ and its product cordial labeling

**Theorem 3.7.5.** The graph $DS(B_{n,n})$ is not product cordial graph.

**Proof.** For the graph $DS(B_{n,n})$, $|V(DS(B_{n,n}))| = 2n + 4$ and $|E(DS(B_{n,n}))| = 4n + 3$. In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $n + 2$ vertices out of $2n + 4$ vertices. The vertices with label 0 will give rise to at least $2n + 3$ edges with label 0 and at most $2n$ edges with label 1 out of total $4n + 3$ edges. Therefore $|e_f(0) - e_f(1)| = 3$. Thus the edge condition for a graph to be product cordial is violated.

Hence the graph $DS(B_{n,n})$ is not product cordial graph.

**Theorem 3.7.6.** The graph $DS(G_n)$ is not product cordial graph.

**Proof.** For the graph $DS(G_n)$, $|V(DS(G_n))| = 2n + 3$ and $|E(DS(G_n))| = 5n$. In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $n + 1$ vertices out of $2n + 3$ vertices. The vertices with label 0 will give rise to at least $3n$ edges with label 0 and at most $2n$ edges with label 1 out of total $5n$ edges. Therefore $|e_f(0) - e_f(1)| = n > 1$. Thus the edge condition for a graph to be product cordial is violated.

Hence the graph $DS(G_n)$ is not product cordial graph.
3.8 Product Cordial Labeling of Some Special Graphs

**Theorem 3.8.1.** Closed helm $CH_n$ is a product cordial graph.

*Proof.* Let $v$ be the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of inner cycle and $u_1, u_2, \ldots, u_n$ be the vertices of outer cycle of $CH_n$. Then $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$.

We define $f : V(CH_n) \rightarrow \{0, 1\}$ as follows.

$$f(v) = 1;$$
$$f(v_i) = 1, \text{ for all } i;$$
$$f(u_i) = 0, \text{ for all } i.$$

In view of the above defined labeling pattern we have

$$v_f(0) = v_f(1) - 1 = n$$
$$e_f(0) = e_f(1) = 2n$$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| = 0$. Hence $CH_n$ is a product cordial graph. ■

**Illustration 3.8.2.** The graph $CH_5$ and its product cordial labeling is shown in Figure 3.26.

**Theorem 3.8.3.** Web graph $Wb_n$ admits product cordial labeling.

*Proof.* Let $v$ be the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of inner cycle, $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ be the vertices of outer cycle and $v_{2n+1}, v_{2n+2}, \ldots, v_{3n}$ be the pendant vertices in $Wb_n$. Then $|V(Wb_n)| = 3n + 1$ and $|E(Wb_n)| = 5n$.

To define $f : V(Wb_n) \rightarrow \{0, 1\}$ we consider following two cases.

**Case 1:** When $n$ is odd.
Figure 3.26: The graph $CH_5$ and its product cordial labeling

$$f(v) = 1;$$
$$f(v_i) = 1, \quad \text{for } 1 \leq i \leq n;$$
$$f(v_{2i}) = 1, \quad \text{for } \left\lceil \frac{n}{2} \right\rceil \leq i \leq n - 1;$$
$$f(v_i) = 0, \quad \text{otherwise}.$$  

In view of the above defined labeling pattern we have

$$v_f(0) = v_f(1) = \frac{3n + 1}{2}$$
$$e_f(0) - 1 = e_f(1) = \frac{5n - 1}{2}$$

**Case 2:** When $n$ is even.

$$f(v) = 1;$$
$$f(v_i) = 1, \quad \text{for } 1 \leq i \leq n;$$
$$f(v_{2i+1}) = 1, \quad \text{for } \frac{n}{2} \leq i \leq n - 1;$$
$$f(v_i) = 0, \quad \text{otherwise}.$$  

In view of the above defined labeling pattern we have

$$v_f(0) = v_f(1) - 1 = \frac{3n}{2}$$
$$e_f(0) = e_f(1) = \frac{5n}{2}$$
Thus in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence $Wbn$ admits product cordial labeling.

**Illustration 3.8.4.** The web graph $Wb_5$ and its product cordial labeling is shown in Figure 3.27.

![Figure 3.27: The web graph $Wb_5$ and its product cordial labeling](image)

**Theorem 3.8.5.** Flower graph $Fl_n$ admits product cordial labeling.

*Proof.* Let $H_n$ be the helm corresponding to $Fl_n$ with $v$ as the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of cycle and $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ be the pendant vertices. Then $|V(Fl_n)| = 2n + 1$ and $|E(Fl_n)| = 4n$.

We define $f : V(Fl_n) \to \{0, 1\}$ as follows.

\[
\begin{align*}
f(v) &= 1; \\
f(v_i) &= 1, \quad \text{for } 1 \leq i \leq n; \\
f(v_i) &= 0, \quad \text{for } n + 1 \leq i \leq 2n.
\end{align*}
\]

In view of the above defined labeling pattern we have

\[
\begin{align*}
v_f(0) &= v_f(1) - 1 = n \\
e_f(0) &= e_f(1) = 2n
\end{align*}
\]
Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence $Fl_n$ admits product cordial labeling.

**Illustration 3.8.6.** The flower graph $Fl_5$ and its product cordial labeling is shown in Figure 3.28.

![Figure 3.28: The flower graph $Fl_5$ and its product cordial labeling](image)

**Theorem 3.8.7.** Gear graph $G_n$ is a product cordial graph for odd $n$ and not a product cordial graph for even $n$.

**Proof.** Let $W_n$ be the wheel with apex vertex $v$ and rim vertices $v_1, v_2, \ldots, v_n$. In order to obtain the gear graph $G_n$ subdivide each rim edge of wheel by the vertices $u_1, u_2, \ldots, u_n$. Where each $u_i$ subdivides the edge $v_iv_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $u_n$ subdivides the edge $v_1v_n$. Then $|V(G_n)| = 2n + 1$ and $|E(G_n)| = 3n$.

To define $f : V(G_n) \to \{0, 1\}$ we consider following two cases.

**Case 1:** When $n$ is odd.

$$f(v) = 1;$$

$$f(v_i) = 1, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor;$$

$$f(v_i) = 0, \quad \text{otherwise;}$$

$$f(u_i) = 1, \quad \text{for } 1 \leq i \leq n + \left\lfloor \frac{n}{2} \right\rfloor;$$

$$f(u_i) = 0, \quad \text{otherwise.}$$

In view of the above defined labeling pattern we have
\[ v_f(0) = v_f(1) - 1 = n \]
\[ e_f(0) = e_f(1) + 1 = \frac{3n+1}{2} \]

Thus we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\).

**Case 2:** When \(n\) is even. In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(n\) vertices out of \(2n+1\) vertices. The vertices with label 0 will give rise to at least \(\frac{3n}{2} + 1\) edges with label 0 and at most \(\frac{3n}{2} - 1\) edge with label 1 out of total \(3n\) edges. Therefore \(|e_f(0) - e_f(1)| = 2\). Thus the edge condition for a graph to be product cordial is violated.

Hence gear graph is a product cordial graph for odd \(n\) and not a product cordial graph for even \(n\). \(\blacksquare\)

**Illustration 3.8.8.** The gear graph \(G_7\) and its product cordial labeling is shown in Figure 3.29.

![Figure 3.29: The gear graph \(G_7\) and its product cordial labeling](image)

**Theorem 3.8.9.** \(CL_n\) is not a product cordial graph.

**Proof.** The circular ladder \(CL_n\) has \(2n\) vertices and \(3n\) edges. In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \(n\) vertices out of total \(2n\) vertices. The vertices with label 0 will give rise to at least \(2n\) edges with label 0 and at most \(n\) edges with label 1 out of total \(3n\) edges. Therefore
Thus the edge condition for a graph to be product cordial is violated. Hence $CL_n$ is not a product cordial graph.

**Theorem 3.8.10.** $M_n$ is not a product cordial graph.

**Proof.** The Möbius ladder $M_n$ has $2n$ vertices and $3n$ edges. To prove the result we consider following two cases.

**Case 1:** When $n = 3$.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to 3 vertices out of total 6 vertices. The vertices with label 0 will give rise to at least 7 edges with label 0 and at most 2 edges with label 1 out of total 9 edges. Therefore $|e_f(0) - e_f(1)| = 5$. Thus the edge condition for a graph to be product cordial is violated.

**Case 2:** When $n \neq 3$.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $n$ vertices out of total $2n$ vertices. The vertices with label 0 will give rise to at least $2n$ edges with label 0 and at most $n$ edges with label 1 out of total $3n$ edges. Therefore $|e_f(0) - e_f(1)| = n$. Thus the edge condition for a graph to be product cordial is violated.

Hence $M_n$ is not a product cordial graph.

**Theorem 3.8.11.** $S(T_n)$ is not a product cordial graph.

**Proof.** The step ladder $S(T_n)$ has $\frac{n^2 + 5n + 2}{2}$ vertices and $n^2 + 3n$ edges. To prove the result we consider following two cases.

**Case 1:** When $\frac{n^2 + 5n + 2}{2}$ is odd.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to $\left\lfloor \frac{n^2 + 5n + 2}{4} \right\rfloor$ vertices out of total $\frac{n^2 + 5n + 2}{2}$ vertices. The vertices with label 0 will give rise to at least $\frac{n^2 + 3n + 4}{2}$ edges with label 0 and at most
\[ \frac{n^2 + 3n - 4}{2} \] edges with label 1 out of total \( n^2 + 3n \) edges. Therefore \( |e_f(0) - e_f(1)| = 4 \). Thus the edge condition for a graph to be product cordial is violated.

**Case 2:** When \( \frac{n^2 + 5n + 2}{2} \) is even.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( \frac{n^2 + 5n + 2}{4} \) vertices out of total \( \frac{n^2 + 5n + 2}{2} \) vertices. The vertices with label 0 will give rise to at least \( \frac{n^2 + 3n + 4}{2} \) edges with label 0 and at most \( \frac{n^2 + 3n - 4}{2} \) edges with label 1 out of total \( n^2 + 3n \) edges. Therefore \( |e_f(0) - e_f(1)| = 4 \). Thus the edge condition for a graph to be product cordial is violated.

\( S(T_n) \) is not a product cordial graph.

**Theorem 3.8.12.** \( H_{n,n} \) is a product cordial graph for \( n = 2 \) and not a product cordial graph for \( n > 2 \).

**Proof.** The graph \( H_{n,n} \) has \( 2n \) vertices and \( \frac{n(n+1)}{2} \) edges. To prove the result we consider following three cases.

**Case 1:** When \( n = 2 \).

The graph \( H_{2,2} = P_2 \), which is product cordial as proved in [53].

**Case 2:** When \( n \) is odd.

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of total \( 2n \) vertices. The vertices with label 0 will give rise to at lest \( \frac{n^2 + 3}{4} \) edges with label 0 and at most \( \frac{n^2 - 1}{4} \) edges with label 1 out of total \( \frac{n(n+1)}{2} \) edges. Therefore \( |e_f(0) - e_f(1)| = \left\lceil \frac{n}{2} \right\rceil \). Thus the edge condition for a graph to be product cordial is violated.

**Case 3:** When \( n \) is even and \( n > 2 \).

In order to satisfy the vertex condition for a graph to be product cordial it is essential to assign label 0 to \( n \) vertices out of total \( 2n \) vertices. The vertices with label 0 will give
rise to at least \( \frac{n^2 + 4}{4} \) edges with label 0 and at most \( \frac{n^2}{4} \) edges with label 1 out of total \( \frac{n(n + 1)}{2} \) edges. Therefore \( |e_f(0) - e_f(1)| = \frac{n}{2} \). Thus the edge condition for a graph to be product cordial is violated.

Hence \( H_{n,n} \) is a product cordial graph for \( n = 2 \) and not a product cordial graph for \( n > 2 \).

**Theorem 3.8.13.** The graph obtained by joining

1. \( G(p,q) \) and \( G'(p,q) \)
2. \( G(p,q) \) and \( G'(p-1,q) \)
3. \( G(p,q) \) and \( G'(p,q-1) \)
4. \( G(p,q) \) and \( G'(p-1,q-1) \)
5. \( G(p-1,q) \) and \( G'(p,q-1) \)

by a path \( P_k \) of arbitrary length admits product cordial labeling where \( G \) and \( G' \) are two arbitrary graphs.

**Proof.** Let \( \{v_1, v_2 \ldots v_n\} \) be the set of vertices of path \( P_k \), \( \{x_1, x_2 \ldots x_n\} \) be the set of vertices of a graph \( G \) and \( \{y_1, y_2 \ldots y_n\} \) be the set of vertices of a graph \( G' \). Let \( G'' \) be the graph obtained by joining \( G \) and \( G' \) by \( P_k \) such that \( x_1 = v_1 \) and \( y_1 = v_n \).

To define \( f : V \to \{0,1\} \) we consider following ten cases.

**Case 1:** When \( G'' \) is obtained by joining \( G(p,q) \) and \( G'(p,q) \) with path \( P_k \) and \( k \equiv 0 \) (mod 2).

The graph \( G'' \) has \( 2p + k - 2 \) vertices and \( 2q + k - 1 \) edges.
\[ f(x_i) = 0, \quad \text{for all } i \]
\[ f(y_i) = 1, \quad \text{for all } i \]
\[ f(v_i) = 0, \quad 2 \leq i \leq \frac{k}{2} \]
\[ f(v_i) = 1, \quad \frac{k}{2} + 1 \leq i \leq k - 1 \]

In view of the above defined labeling pattern we have,

\[ v_f(0) = v_f(1) = p + \frac{k}{2} - 1 \]
\[ e_f(0) = e_f(1) + 1 = q + \frac{k}{2} \]

Case 2: When \( G'' \) is obtained by joining \( G(p, q) \) and \( G'(p, q) \) with path \( P_k \) and \( k \equiv 1 \) (mod 2).

The graph \( G'' \) has \( 2p + k - 2 \) vertices and \( 2q + k - 1 \) edges.

\[ f(x_i) = 0, \quad \text{for all } i \]
\[ f(y_i) = 1, \quad \text{for all } i \]
\[ f(v_i) = 0, \quad 2 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \]
\[ f(v_i) = 1, \quad \left\lceil \frac{k}{2} \right\rceil + 1 \leq i \leq k - 1 \]

In view of the above defined labeling pattern we have,

\[ v_f(0) + 1 = v_f(1) = p + \frac{k - 1}{2} \]
\[ e_f(0) = e_f(1) = q + \frac{k - 1}{2} \]

Case 3: When \( G'' \) is obtained by joining \( G(p, q) \) and \( G'(p - 1, q) \) with path \( P_k \) and \( k \equiv 0 \) (mod 2).

The graph \( G'' \) has \( 2p + k - 3 \) vertices and \( 2q + k - 1 \) edges.
Chapter 3. Product Cordial Labeling of Graphs

\[ f(x_i) = 0, \quad \text{for all } i \]
\[ f(y_i) = 1, \quad \text{for all } i \]
\[ f(v_i) = 0, \quad 2 \leq i \leq \frac{k}{2} \]
\[ f(v_i) = 1, \quad \frac{k}{2} + 1 \leq i \leq k - 1 \]

In view of the above defined labeling pattern we have,

\[ v_f(0) = v_f(1) + 1 = p + \frac{k}{2} - 1 \]
\[ e_f(0) = e_f(1) + 1 = q + \frac{k}{2} \]

**Case 4:** When \( G'' \) is obtained by joining \( G(p, q) \) and \( G'(p - 1, q) \) with path \( P_k \) and \( k \equiv 1 \) (mod 2).

The graph \( G'' \) has \( 2p + k - 3 \) vertices and \( 2q + k - 1 \) edges.

\[ f(x_i) = 0, \quad \text{for all } i \]
\[ f(y_i) = 1, \quad \text{for all } i \]
\[ f(v_i) = 0, \quad 2 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \]
\[ f(v_i) = 1, \quad \left\lceil \frac{k}{2} \right\rceil + 1 \leq i \leq k - 1 \]

In view of the above defined labeling pattern we have,

\[ v_f(0) = v_f(1) = p + \frac{k - 3}{2} \]
\[ e_f(0) = e_f(1) = q + \frac{k - 1}{2} \]

**Case 5:** When \( G'' \) is obtained by joining \( G(p, q) \) and \( G'(p, q - 1) \) with path \( P_k \) and \( k \equiv 0 \) (mod 2).

The graph \( G'' \) has \( 2p + k - 2 \) vertices and \( 2q + k - 2 \) edges.
\[ f(x_i) = 1, \quad \text{for all } i \]
\[ f(y_i) = 0, \quad \text{for all } i \]
\[ f(v_i) = 1, \quad 2 \leq i \leq \frac{k}{2} \]
\[ f(v_i) = 0, \quad \frac{k}{2} + 1 \leq i \leq k - 1 \]

In view of the above defined labeling pattern we have,

\[ v_f(0) = v_f(1) = p + \frac{k}{2} - 1 \]
\[ e_f(0) = e_f(1) = q + \frac{k}{2} - 1 \]

**Case 6:** When \( G'' \) is obtained by joining \( G(p, q) \) and \( G'(p, q - 1) \) with path \( P_k \) and \( k \equiv 1 \) (mod 2).

The graph \( G'' \) has \( 2p + k - 2 \) vertices and \( 2q + k - 2 \) edges.

\[ f(x_i) = 1, \quad \text{for all } i \]
\[ f(y_i) = 0, \quad \text{for all } i \]
\[ f(v_i) = 1, \quad 2 \leq i \leq \left\lceil \frac{k}{2} \right\rceil \]
\[ f(v_i) = 0, \quad \left\lceil \frac{k}{2} \right\rceil + 1 \leq i \leq k - 1 \]

In view of the above defined labeling pattern we have,

\[ v_f(0) = v_f(1) - 1 = p + \frac{k - 3}{2} \]
\[ e_f(0) + 1 = e_f(1) = q + \frac{k - 1}{2} \]

**Case 7:** When \( G'' \) is obtained by joining \( G(p, q) \) and \( G'(p - 1, q - 1) \) with path \( P_k \) and \( k \equiv 0 \) (mod 2).

The graph \( G'' \) has \( 2p + k - 3 \) vertices and \( 2q + k - 2 \) edges.
\[ f(x_i) = 1, \text{ for all } i \]
\[ f(y_i) = 0, \text{ for all } i \]
\[ f(v_i) = 1, \quad 2 \leq i \leq \frac{k}{2} \]
\[ f(v_i) = 0, \quad \frac{k}{2} + 1 \leq i \leq k - 1 \]

In view of the above defined labeling pattern we have,
\[ v_f(0) = v_f(1) - 1 = p + \frac{k}{2} - 2 \]
\[ e_f(0) = e_f(1) = q + \frac{k}{2} - 1 \]

**Case 8:** When \( G'' \) is obtained by joining \( G(p, q) \) and \( G'(p - 1, q - 1) \) with path \( P_k \) and \( k \equiv 1 \pmod{2} \).

The graph \( G'' \) has \( 2p + k - 3 \) vertices and \( 2q + k - 2 \) edges.

\[ f(x_i) = 1, \text{ for all } i \]
\[ f(y_i) = 0, \text{ for all } i \]
\[ f(v_i) = 1, \quad 2 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \]
\[ f(v_i) = 0, \quad \left\lfloor \frac{k}{2} \right\rfloor + 1 \leq i \leq k - 1 \]

In view of the above defined labeling pattern we have,
\[ v_f(0) = v_f(1) = p + \frac{k - 3}{2} \]
\[ e_f(0) = e_f(1) + 1 = q + \frac{k - 1}{2} \]

**Case 9:** When \( G'' \) is obtained by joining \( G(p, q - 1) \) and \( G'(p - 1, q) \) with path \( P_k \) and \( k \equiv 0 \pmod{2} \).

The graph \( G'' \) has \( 2p + k - 3 \) vertices and \( 2q + k - 2 \) edges.
$f(x_i) = 0, \text{ for all } i$

$f(y_i) = 1, \text{ for all } i$

$f(v_i) = 0, \quad 2 \leq i \leq \frac{k}{2}$

$f(v_i) = 1, \quad \frac{k}{2} + 1 \leq i \leq k - 1$

In view of the above defined labeling pattern we have,

$v_f(0) = v_f(1) + 1 = p + \frac{k}{2} - 1$

$e_f(0) = e_f(1) = q + \frac{k}{2} - 1$

**Case 10:** When $G''$ is obtained by joining $G(p, q - 1)$ and $G'(p - 1, q)$ with path $P_k$ and $k \equiv 1 \text{ (mod 2)}$.

The graph $G''$ has $2p + k - 3$ vertices and $2q + k - 2$ edges.

$f(x_i) = 0, \text{ for all } i$

$f(y_i) = 1, \text{ for all } i$

$f(v_i) = 0, \quad 2 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$

$f(v_i) = 1, \quad \left\lfloor \frac{k}{2} \right\rfloor + 1 \leq i \leq k - 1$

In view of the above defined labeling pattern we have,

$v_f(0) = v_f(1) = p + \frac{k - 3}{2}$

$e_f(0) = e_f(1) - 1 = q + \frac{k - 3}{2}$

Thus from the above discussion we have the required result.

**Illustration 3.8.14.** The graph obtained by joining $G = C_5$ and $G' = W_3$ with path $P_3$ and its product cordial labeling is shown in Figure 3.30.
Theorem 3.8.15. Graph $< G_1 \triangle G_2 >$ is product cordial.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of $G_1$ and $u_1, u_2, \ldots, u_n$ be the vertices of $G_2$. Let $G$ be the graph $< G_1 \triangle G_2 >$. Then $|V(G)| = 2p + 1$ and $|E(G)| = 2q + 3$.

To define $f : V(G) \rightarrow \{0, 1\}$ we have

\[
\begin{align*}
  f(v_i) &= 1, \quad \text{for all } i \\
  f(u_i) &= 0, \quad \text{for all } i \\
  f(v') &= 1
\end{align*}
\]

In view of the above defined labeling pattern we have

\[
\begin{align*}
  v_f(0) &= v_f(1) - 1 = p \\
  e_f(0) - 1 &= e_f(1) = q + 1
\end{align*}
\]

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence the graph $< G_1 \triangle G_2 >$ is product cordial. \qed

Illustration 3.8.16. The graph $< K_{1,5} \triangle K_{1,5} >$ and its product cordial labeling is shown in Figure 3.31.
3.9 Concluding Remarks and Further Scope of Research

The product cordial labeling is a variant of cordial labeling where edge labels are induced by product of vertex labels. We have observed that product cordial labeling and cordial labeling of a graph are two independent concepts. A graph may possess one or both of these labelings or neither as exhibited below.

- Trees are cordial as well as product cordial.
- The complete bipartite graph $K_{m,n}$ for $m,n > 2$ is cordial but not product cordial.
- The friendship graph $F_n$ for $n \equiv 2(\text{mod } 4)$ is not cordial but it is product cordial.
- The complete graph $K_n$ for $n > 4$ is neither cordial nor product cordial.

To investigate necessary and sufficient condition for a graph to admit a product cordial labeling, some characterizations for product cordial labeling and to identify linkages between cordial labeling and product cordial labeling are some open areas of research.