CHAPTER - VI

PERSISTENCE
**PERSISTENCE**

In the present chapter the criteria for uniform persistence of the model will be obtained. First of all the conditions for strong persistence of the system will be derived. After this the conditions will be given under which the system shows uniform persistence.

6.1 Uniform Persistence of a Simple Food Chain

The system (4.4) forms a simple food chain of length $i$ in $\mathbb{R}^i_{x_1, x_2, \ldots, x_i}$

**Theorem 6.1**

If the hypotheses $L(1, 2, 3, 4, 5, 6, 7)$ hold, then the system (4.4) is uniformly persistent.

**Proof**

The proof of the above theorem follows from the result of Freedman and So(1985).

6.2 Facultative Mutualism with the Prey

Now the criteria for uniform persistence of system (3.1) in case of facultative mutualism between $u$ and $x_1$ will be discussed. For this the Lemma required is given below.

**Lemma 6.1**

**Butler McGehee Lemma**

Let $P$ be an isolated hyperbolic equilibrium in the omega limit set $\Omega(x)$ of an orbit $O(x)$. Then either $\Omega(x) = P$ or there exists points $Q^+$ and $Q^-$ in $\Omega(x)$ with $Q^+ \in W^s(P)$ and $Q^- \in W^u(P)$. Where $W^s(P)$ and $W^u(P)$ are stable and unstable manifolds respectively of $P$.

**Proof**

**Remark 6.1**

Similar arguments can be made for every isolated invariant set where the stable and unstable manifolds can be defined. In particular the argument will work for an isolated closed orbit in one of the coordinate planes.

The system (3.1) exhibits facultative mutualism between \( u \) and \( x_1 \) whenever the hypotheses \( H \) (1-4), \( G \) (1-2), \( P \) (1-3), \( S \) (1-3) hold.

**Theorem 6.2**

Let the hypotheses \( H(1, 2, 5, 6, 7), G(1, 2), P(1), S(1) \) hold, then the system (3.1) is strongly persistent.

**Proof**

Let \( x \in R^*_{x_0, x_1, x_2, \ldots, x_n} \). First of all it will be shown that \( E_0, E_k, E_j, E_{j, 1} \notin \Omega(x) \) for \( 1 \leq j \leq n-1, 1 \leq k \leq n \). Suppose \( E_0(0, 0, \ldots 0) \in \Omega(x) \).

Since \( E_0 \) is a saddle point \( \{E_0\} \neq \Omega(x) \), by Butler McGehee Lemma \( \exists Q_0 \in W^s(E_0) \setminus \{E_0\} \) such that \( Q_0 \in \Omega(x) \). It means the whole orbit \( O(Q_0) \) through \( Q_0 \) will belong to \( \Omega(x) \).

\[
W^s(E_0) = R^*_{x_2, x_3, \ldots, x_n}
\]

All the orbits in the given stable set are unbounded. Which contradicts the boundedness property of \( \Omega(x) \). Thus \( E_0 \notin \Omega(x) \).

Suppose \( E_{0,1} \in \Omega(x) \). Being a hyperbolic saddle point, we have \( \{E_{0,1}\} \neq \Omega(x) \).

By Butler McGehee Lemma \( \exists Q_{0,1} \in W^s(E_{0,1}) \setminus \{E_{0,1}\} \) such that \( Q_{0,1} \in \Omega(x) \).

But \( W^s(E_{0,1}) = \{x \mid x \in R^*_{x_1, x_2, x_3, \ldots, x_k}, \}

for all \( \{j, j_2, \ldots, j_k\} \subseteq \{3, 4, \ldots, n\} \)

The closure of the orbit through \( Q_{0,1} \) either contain \( E_0 \) as its \( \alpha \) limit point or is unbounded. Which leads to contradiction i.e. \( E_{0,1} \notin \Omega(x) \).
Now let $E_{1,0} \in \Omega(x)$.

$E_{1,0}$ is also a hyperbolic saddle point, so $\exists Q_{1,0} \in \Omega(x)$ such that $Q_{1,0} \in W^s(E_{1,0}) \setminus \{E_{1,0}\}$

where $W^s(E_{1,0}) = \{x \mid x \in R_{x_1, x_2, ..., x_n}^+ \}$

for all $\{j_1, j_2, ..., j_k\} \subseteq \{2, 3, 4, ..., n\}$

It is clear that all the orbits in the stable manifold of $E_{1,0}$ are either unbounded or contain $E_0$ as its \(\alpha\) limit point. Which are contradictions. Thus $E_{1,0} \not\in \Omega(x)$.

Further we see that all the remaining boundary equilibria i.e. $E_{j,0}$ and $E_{k,1}$, $2 \leq j \leq n$, $1 \leq k \leq n-1$ are hyperbolic saddle equilibria. So they can not be equal to $\Omega(x)$ and the orbits in their stable manifolds are either unbounded or contain any of the lower dimensional equilibrium points as its \(\alpha\) limit point. Both are contradictions. It means the boundary equilibria do not belong to omega limit set of $x$. Now we claim that no point of boundary belongs to $\Omega(x)$.

Now suppose $x_0 \in R_{x_1, x_2, ..., x_n}^+$ or $x_0 \in R_{x_1, x_2, ..., x_n}$, where $\{j_1, j_2, ..., j_m\} \subseteq \{k+2, k+3, ..., n\}$,

then either $\Omega(x)$ is unbounded or $E_{k,0} \in \Omega(x)$.

Similarly it can be shown easily that if $x_0 \in R_{u, x_1, x_2, ..., x_n}^+$ or $x_0 \in R_{u, x_1, x_2, ..., x_n}$, where $\{j_1, j_2, ..., j_m\} \subseteq \{k+2, k+3, ..., n\}$,

then $\Omega(x)$ is unbounded or $E_{k,1} \in \Omega(x)$.

Both are contradictions.

Hence the system (3.1) is strongly persistent.

**Theorem 6.3**

Let the hypotheses $H(1, 2, 3, 4, 6, 7)$, $G(1, 2, 3, 4)$, $P(1, 2, 3)$, $S(1, 2, 3)$ hold them the system (3.1) is uniformly persistent.
**Proof**

As the system (3.1) has a compact region of attraction. It follows from the Theorem 3.2 that the system is dissipative. Theorem 6.2 proves it to be strongly persistent. Using hypotheses (H6) and (H7) it can be easily shown to have isolated and acyclic boundary. Thus, according to Theorem 2.13 the system is uniformly persistent.

Now the criteria for uniform persistence of (3.1) will be given when there exist periodic solutions in positive \( x_1-x_2 \) plane.

(H8) Let the equilibria \( E_{j,0} \) and \( E_{k,1} \), \( 1 \leq j \leq n, j \neq 2, 1 \leq k \leq n-1 \) exist and be globally stable in their respective subsystems and the equilibrium in \( x_1-x_2 \) plane be unique.

**Theorem 6.4**

Under the hypotheses H(1-4), H(7-8), G(1-4), P(1-3) and S(1-3) for system (3.1), if there exist a finite number of periodic solutions \( x_1 = \phi_i(t), x_2 = \Psi_i(t), 1 \leq i \leq k \) in \( R_{x_1,x_2}^* \). Then the system in \( R_{u_1,x_1,x_2,x_n}^* \) is uniformly persistent provided \( \beta_3(2) > 0 \) for each periodic solution of period \( T \) in \( R_{x_1,x_2}^* \).

\[
m_i = -s_3(0,0) + \frac{1}{T} \int_0^T c_3(0) p_2(0,\Psi_i(t)) \, dt > 0, \quad 1 \leq i \leq k,
\]

and

\[
\beta_3(2) = -s_3(0,0) + c_3(0) p_2(0,x_2),
\]

where \( \beta_3(2) \) is eigenvalue of \( E_{2,0} \) along \( x_3 \).

**Proof**

For any \( x \in R_{u_1,x_1,x_2,x_n}^* \) let \( O(x) \) denotes the orbit through \( x \) and \( \Omega(x) \) be its omega limit set. By similar arguments as given in Theorem 6.2 it can be shown that \( E_0,E_{1,0} \) and \( E_{0,1} \notin \Omega(x) \).
Now it will be shown that no periodic orbit in $R^*_x \times 2$ or $E_{2,0}$ belong to $\Omega(x)$. Let $\overline{p}_i$, $1 \leq i \leq k$ denote the closed orbits in $R^*_x \times 2$, which has its periodic solutions $(\phi_i(t), \Psi_i(t))$. Where $\overline{p}_i$ lies inside $\overline{p}_{i-1}$.

The variational matrix $V_i(0, \phi_i(t), \Psi_i(t), 0, \ldots 0)$ corresponding to orbit $p_i$ is given by table 6.1.

Computing the fundamental matrix of the linear periodic system

$$X' = V_i(t)X, \quad X(0) = I \tag{6.1}$$

We find that its characteristic multiplier in the $x_3$ direction is $e^{m_1 \tau}$.

Suppose $\overline{p}_i$ lies in $\Omega(x)$. Then from table (6.1) its Floquet multiplier in $x_3$ direction is greater than one and therefore by Theorem 2.4 and by Butler-McGehee Lemma there exists a point $Q_1 \in W^s(\overline{p}_i) \cap \Omega(x)$. If $Q_1$ lies outside $\overline{p}_1$ then $O(Q_1)$ lies in $\Omega(x)$ which is unbounded. If $\overline{p}_i$ is stable from inside then $\alpha(Q_1)$ must be $\overline{p}_2$ and it is also contained in $\Omega(x)$. If $\overline{p}_2$ is unstable from inside as well, there is a contradiction. For a repeller cannot be in $\Omega(x)$. If $\overline{p}_2$ is stable from the inside $\exists Q_2$ inside $\overline{p}_2$ such that $Q_2 \in W^s(\overline{p}_2) \cap \Omega(x)$. Then $\overline{p}_3 \subset \Omega(x)$. Continuing this argument it is concluded that either $\overline{p}_i$ is unstable from both sides or it is stable from inside and $\exists Q_k \in W^s(\overline{p}_k) \cap \Omega(x)$.

If such a $Q_k$ exists then $E_{2,0}$ must be a repeller and $\alpha(Q_k) = E_{2,0}$ lies in $\Omega(x)$ which is again a contradiction. Thus $\overline{p}_i$ does not lie in $\Omega(x)$. By similar arguments it can be proved that no $\overline{p}_i$ lies in $\Omega(x)$.

Further by arguments similar to those in Theorem 6.2 no boundary equilibria lies in $\Omega(x)$, so the system is strongly persistent.

The system is dissipative too, as it has a compact region of attraction. It is clear from hypotheses (H7) and (H8) that it has isolated and acyclic boundary. Thus according to the result of Theorem 2.13 the system is uniformly persistent.
6.3 Facultative Mutualism with an Intermediate Prey/Predator $x_j$, $j \geq 2$

**Theorem 6.5**

If the hypotheses $H(1,2,3**,4**,6,7)$, $G(1,2,3**)$ $S(1,2**,3)$, $P(1,2**,3**)$ hold, then the system (3.1) is uniformly persistent.

**Proof**

The proof of this theorem is similar to that of Theorem 6.3.

The criterion for uniform persistence of system (3.1) can be derived in case of periodic solutions in $x_1$-$x_2$ plane, by similar explanations as given in Theorem 6.4.

**Theorem 6.6**

Under the hypotheses $H(1,2,3**,4**,7,8)$, $G(1,2,3**)$, $P(1,2**,3**)$, $S(1,2**,3)$ for system (3.1), if there exist a finite number of periodic solutions $x_1 = \phi(t), x_2 = \psi(t)$, $1 \leq i \leq k$ in $R^+_{x_1,x_2}$. Then the system in $R^+_{u,x_1,x_2,x_3}$ is uniformly persistent provided $\beta_3(2) > 0$ for each periodic solution of period $T$ in $R^+_{x_1,x_2}$,

$$m_1 = -s_3(0,0) + \frac{1}{T} \int_0^T c_3(0)p_2(0,\psi(t))\,dt > 0, 1 \leq i \leq k,$$

and

$$\beta_3(2) = -s_3(0,0) + c_3(0)p_2(0,x_{22}),$$

where $\beta_3(2)$ is eigenvalue of $E_{2,0}(0,x_{21},x_{22},0,0,...,0)$ along $x_3$.

6.4 Obligate Mutualism with an Intermediate Prey/Predator $x_j$, $j \geq 2$

**Theorem 6.7**

Under the hypotheses $H(1,2,3*,4*,6,7)$ $G(1,2,3*)$, $P(1,2*,3*)$, $S(1,2*,3*,4*)$ the system (3.1) is uniformly persistent.
Proof

The proof of the above theorem is similar to that of Theorem 6.3.

By similar arguments as given in Theorem 6.4 it can be shown that the system (3.1) uniformly persists in case of periodic solutions in $x_1-x_2$ plane. Which is given by the following theorem.

**Theorem 6.8**

Under the hypotheses $H(1,2,3^*,4^*,7,8)$, $G(1,2,3^*)$, $P(1,2^*,3^*)$, $S(1,2^*,3^*,4^*)$ for system (3.1) if there exist a finite number of periodic solutions $x_i = \hat{\phi}(t), x_2 = \hat{\psi}(t), 1 \leq i \leq k$ in $R_{x_1,x_2}^*$. Then the system in $R^*_{x_1,x_2}$ is uniformly persistent provided $\hat{\beta}_3(2) > 0$ for each periodic solution of period $T$ in $R^*_{x_1,x_2}$.

$$m_i = -s_3(0,0) + \frac{1}{T} \int_0^T c_3(0) p_2(0,\hat{\psi}(t)) \, dt > 0, \quad 1 \leq i \leq k,$$

and

$$\hat{\beta}_3(2) = -s_3(0,0) + c_3(0) p_2(0,\hat{x}_2),$$

where $\hat{\beta}_3(2)$ is eigenvalue of $E_{2,0}(0,\hat{x}_2,0,\ldots,0)$ along $x_3$.

**6.5 Obligated Mutualism with the Top Predator**

**Theorem 6.9**

Under the hypotheses $H(1,2,3',4',6,7)$, $G(1,2,3')$, $P(1,2')$, $S(1,2',3',4')$ the system (3.1) is uniformly persistent.

Proof

The proof of this theorem is similar to that of Theorem 6.3.

The result below gives the conditions for uniform persistence of system 3.1 in case of existence of periodic solutions in $x_1-x_2$ plane.
**Theorem 6.10**

Under the hypotheses $H(1,2,3',4',7,8)$, $G(1,2,3')$, $P(1,2')$, $S(1,2',3',4')$ for system (3.1), if there exist a finite number of periodic solutions $x_1 = \bar{\phi}(t), x_2 = \bar{\psi}(t), 1 \leq i \leq k$ in $R_{x_1,x_2}^+$. Then the system is uniformly persistent provided $\tilde{\beta}_3(2) > 0$ for each periodic solution of period $T$ in $R_{x_1,x_2}^+$,

$$m_i = -s_3(0,0) + \frac{1}{T} \int_0^T c_3(0) p_2(0, \bar{\psi}_1(t)) \, dt > 0, \ 1 \leq i \leq k,$$

and

$$\tilde{\beta}_3(2) = -s_3(0,0) + c_3(0) p_2(0, \bar{x}_2),$$

where $\tilde{\beta}_3(2)$ is eigenvalue of $E_{2,0}(0, \bar{x}_2, \bar{x}_2, \ldots, 0)$ along $x_3$.

**Proof**

The proof of this theorem is similar to that of Theorem 6.4.
EXAMPLE - 1:

Facultative Mutualism with the Prey

Consider the system

\[ u' = u(l_1 + \frac{x_1 + l_2}{x_1 + l_3}m_1 - u), \]
\[ x_1' = \alpha x_1(k_0 + m_2(\frac{u + l_4}{u + l_5}) - x_1) \frac{m_3 x_1 x_2}{x_1 + m_4}, \]
\[ x_2' = x_2[-s_2(x_2 + l_6) + \frac{c_2 m_3 x_1}{x_1 + m_4}] \frac{m_5 x_1 x_3}{x_2 + m_6}, \]
\[ x_3' = x_3[-s_3(x_3 + l_7) + \frac{c_3 m_5 x_2}{x_2 + m_6}]. \]

Let

\[ l_1 = 1, \quad l_2 = l_4 = 9, \quad l_3 = l_5 = 10, \quad l_6 = l_7 = 1, \quad m_1 = 1, \quad m_2 = .2, \quad m_3 = 1, \quad m_4 = 20, \]
\[ m_5 = 1, \quad m_6 = 25, \quad c_2 = c_3 = .8, \quad \alpha = .02, \quad k_0 = 1, \quad s_2 = s_3 = .01. \]

The above system has the boundary equilibria \( E_0(0,0,0,0) \), \( E_{0,1}(1.9,0,0,0) \), \( E_{1,0}(0,1.18,0,0) \), \( E_{1,1}(1.91,1.183,0,0) \), \( E_{2,0}(0,.341,.341,0) \), \( E_{2,1}(1.90,.408,.317,0) \), \( E_{3,0}(0,.375,.328,.036) \) and the interior equilibrium \( E^*(1.904,.377,.329,.038) \).

We construct the Liapunov functions for various boundary equilibrium points to check their global stability in their respective sub-domains by using the formulas,

\[ V(u, x_1, x_2, \ldots x_i) = u - \bar{u} - u \ln\left(\frac{u}{\bar{u}}\right) + \sum_{j=1}^{i} (x_j - \bar{x}_j \ln(\frac{x_j}{\bar{x}_j})), \]  \hspace{1cm} (6.3)

\[ V(x_1, x_2, \ldots x_i) = \sum_{j=1}^{i} x_j - \hat{x}_j - \hat{x}_j \ln(\frac{x_j}{\hat{x}_j}), \]  \hspace{1cm} (6.4)

whichever is suitable.

The Liapunov functions are converted into the form

\[ V(u, x_1, x_2, \ldots x_i) = -\mathcal{F}^T \mathcal{F}, \]  \hspace{1cm} (6.5)

where

\[ \mathcal{F} = (u - \bar{u}, x_1 - \bar{x}_1, \ldots x_i - \bar{x}_i). \]
and
\[ V(x_1, x_2, \ldots, x_i) = -T F T^T, \]  \hfill (6.6)

where
\[ T = (x_1 - \hat{x}_1, \ldots, x_i - \hat{x}_i, \ldots, x_j - \hat{x}_j). \]

Here \((\bar{u}, \bar{x}_1, \ldots, \bar{x}_j)\) and \((\hat{x}_1, \ldots, \hat{x}_2, \ldots, \hat{x}_i)\) are the equilibrium points of the boundary and \(F\) is a symmetric matrix. The minimum values of the principal minors of \(F\) are found to be positive in their respective regions of attractions, for each equilibrium point. Thus according to Theorem 5.5 and Theorem 5.6 the equilibrium points are globally stable in the positive orthants of their respective sub-domains. Similarly a Liapunov function \(V(u, x_1, x_2, x_3)\) is constructed for the interior equilibrium \(E^*\) and further converted into the form \(-3F3^T\).

The region of attraction for the system (6.2) is contained in the set \(A\).
\[ A = \{ (u, x_1, x_2, x_3) \mid 0 \leq u \leq 2, 0 \leq x_1 \leq 1.2, 0 \leq x_2 \leq 3.26, 0 \leq x_3 \leq 4.45 \} \]

The minimum values of the principal minors of \(F\) in the set \(A\) are given as
\[ F_1 = 1, F_2 \geq 0.0118, F_3 \geq 0.00008, F_4 \geq 0.000000152. \]

As the minimum values of principal minors are all positive, according to Theorem 5.6 the interior equilibrium is globally stable in \(R^*_u, x_1, x_2, x_3\).

Computing the variational matrix for each boundary equilibrium point and finding their eigenvalues we find that all the boundary equilibria are hyperbolic saddle points. In this way all the conditions of Theorem 6.3 are satisfied and the system (6.2) is proved to be a uniformly persistent system.
EXAMPLE - II:

Facultative Mutualism with the First Predator

Consider the system

\[ u' = u(l_1 + \frac{x_2 + l_2}{x_2 + l_3}m_1 - u), \]
\[ x_1' = \alpha x_1(k_0 - x_1) - \frac{m_2x_1}{x_1 + m_3}x_2, \]
\[ x_2' = x_2[-s_2(x_2 + l_0 - m_4\frac{u + l_4}{u + l_5}) + \frac{c_2m_2x_1}{x_1 + m_3} - \frac{m_5x_2x_3}{x_2 + m_6}, \]
\[ x_3' = x_3[-s_3(x_3 + l_7) + \frac{c_3m_5x_2}{x_2 + m_6}. \]

Let

\[ l_1=1, l_2=l_4=9, l_3=l_5=10, l_6=2, l_7=1, m_1=m_2=1, m_3=20, m_4=1, m_5=1, m_6=60, s_2=s_3=.01, \alpha=.2, k_0=1. \]

The above system has the boundary equilibria \( E_0(0,0,0,0), \)
\( E_{0,1}(1.9,0,0,0), \) \( E_{1,0}(0,1,0,0), \) \( E_{1,1}(1.9,1,0,0), \)
\( E_{2,0}(0,.654,1.431,0), \) \( E_{2,1}(1.91,.65,1.44,0), \)
\( E_{3,0}(0,.739,1.08,.41) \) and the interior equilibrium \( E'(1.91,.739,1.08,.418). \)

The Liapunov functions for various boundary equilibria are constructed using the equations, 6.3 and 6.4 given in the previous example.

The Liapunov functions are converted into the form \(-3F3^T\) or \(-TFT^T\) using equations 6.5 and 6.6, whichever is suitable, and a matrix \( F \) is computed in each case. The minimum values of the principal minors of \( F \) are found to be positive in their respective regions of attraction for each equilibrium point, showing (according to Theorems 5.5 and 5.6) that the equilibrium points are globally stable in the positive orthants of their respective sub-domains. Similarly, a symmetric matrix \( F \) is computed from the Liapunov function \( V(u,x_1,x_2,x) \) for the interior equilibrium point using equation 6.3.
The region of attraction for the given system is contained in the set \( A \) given below.

\[ A = \{ (u, x_1, x_2, x_3) \mid 0 \leq u \leq 2, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 16.8, 0 \leq x_3 \leq 26.24 \} \]

The minimum values of the principal minors of the matrix \( F \) in the set \( A \) are given as

\[ F_1 = 1, \quad F_2 \geq 0.0158, \quad F_3 \geq 0.000376, \quad F_4 \geq 0.0000018. \]

Thus according to Theorem 5.6 the interior equilibrium point is globally stable in \( \mathbb{R}^4 \).

Computing the variational matrix for each boundary equilibrium point and finding their eigenvalues we find that all the boundary equilibria are hyperbolic saddle points in \( \mathbb{R}^4 \).

In this way all the conditions of Theorem 6.5 are satisfied and the system (6.7) is proved to be a uniformly persistent system.
EXAMPLE - III:
Obligate Mutualism with the First Predator

Consider the system

\[
\begin{align*}
    u' &= u(l_1 + \left(\frac{x_2 + l_2}{x_2 + l_3}\right)m_1 - u), \\
    x'_1 &= \alpha x_1(k_0 - x_1) - \frac{m_2 u}{u + m_4} \times \frac{x_1}{x_2} x_2, \\
    x'_2 &= x_2[-s_2(x_2 + l_4) + \frac{c_2 m_2 u}{u + m_4} \times \frac{x_1}{x_1 + m_3} - \frac{x_2 x_3}{x_2 + m_5}], \\
    x'_3 &= x_3[-s_3(x_3 + l_5) + \frac{c_3 x_2}{x_2 + m_5}].
\end{align*}
\]

Let

\[
\begin{align*}
    l_1 &= .8, \quad l_2 = 9, \quad l_3 = 10, \quad l_4 = l_5 = 1, \quad m_1 = .2, \quad m_2 = m_4 = 1, \quad m_3 = 20, \quad m_5 = 35, \\
    \alpha &= .05, \quad k_0 = .9, \quad s_2 = s_3 = .01, \quad c_2 = .8, \quad c_3 = 1.
\end{align*}
\]

The above system has the boundary equilibria \(E_0(0,0,0,0), E_{1,0}(.98,0,0,0), E_{0,1}(0,.9,0,0), E_{1,1}(.98,.9,0,0), E_{1,2}(.98,.72,.38,0),\) and the interior equilibrium \(E'(98,.73,.36,.012).\)

We construct the Liapunov function \(V(u,x_1)\) and \(V(u,x_1,x_2)\) using equation 6.3, to show that the boundary equilibria of two and three dimensions are globally stable in their respective sub-domains.

Further the Liapunov functions are converted into the form 

\[-3F^T \dot{F}\] using the equation 6.5. The minimum values of the principal minors of \(F\) are found to be positive in their respective regions of attractions, for each equilibrium point. It proves the Liapunov functions to be negative definite and equilibrium points to be globally stable in the positive orthants of their respective sub-domains.

Similarly a Liapunov function \(V(u,x_1,x_2,x_3)\) is constructed for the interior equilibrium and further a symmetric matrix \(F\) is computed using equation 6.5. The region of attraction for the system in \(R^+_{u,x_1,x_2,x_3}\) is contained the set \(A\) given as
A=\{(u,x_1,x_2,x_3) | 0 \leq u \leq 1, 0 \leq x_1 \leq .9, 0 \leq x_2 \leq 3.96, 0 \leq x_3 \leq 6.48\}

The minimum values of the principal minors of F in the set A are given below.

F_1 = 1, F_2 \geq .03029, F_3 \geq .00128, F_4 \geq .000124.

According to Theorem 5.6 the interior equilibrium is globally stable in \( R_{u,x_1,x_2,x_3} \). Further, computing the variational matrix for each equilibrium point, it can be easily shown that all the boundary equilibria are hyperbolic saddle points in \( R_{u,x_1,x_2,x_3}^+ \). Thus all the conditions of Theorem 6.7 are satisfied. Hence the system (6.8) is uniformly persistent.
EXAMPLE - IV:

Obligate Mutualism with the Top Predator

Consider the system

\[ u' = u\left[l + m_1\left(\frac{x_3 + l_3}{x_3 + l_2}\right) - u\right], \]
\[ x_1' = \alpha x_1[k_0 - x_1] - \frac{m_2 x_1 x_2}{x_1 + m_3}, \]
\[ x_2' = x_2[-s_2(x_2 + l_3) + \frac{c_2 m_2 x_1}{x_1 + m_3}] - \frac{m_4 x_2 u x_3}{(x_2 + m_5)(u + m_6)}, \]
\[ x_3' = x_3[-s_3(x_3 + l_5) + \frac{c_3 m_4 x_2}{x_2 + m_5} x_1 - \frac{u}{(u + m_6)}]. \]  

(6.9)

Let

\[ l_1 = 9.8, \quad l_2 = 9, \quad l_3 = 10, \quad l_4 = 0.333, \quad l_5 = 0.5, \quad m_1 = 0.2, \quad m_2 = 1, \quad m_3 = 20, \]
\[ m_4 = 3, \quad m_5 = 60, \quad m_6 = 10, \quad s_2 = 0.03, \quad s_3 = 0.02, \quad k_0 = 0.9, \quad \alpha = 0.2, \quad c_2 = 0.8, \quad c_3 = 0.7. \]

The above system has the boundary equilibria \( E_0(0, 0, 0, 0) \), \( E_0, I(9.98, 0, 0, 0) \), \( E_{1, 0}(0.9, 9, 0, 0) \), \( E_{1, 1}(9.98, 9, 0, 0) \), \( E_{2, 1}(9.98, 761, 577, 0) \), and the interior equilibrium \( E^*(9.98, 753, 611, 0.0284) \).

We construct the Liapunov functions \( V(u, x_1) \), \( V(u, x_1, x_2) \) for the boundary equilibria using the equation 6.3. The functions obtained are converted into the form \(-3F^T\) as given by the equation 6.5.

The minimum values of the principal minors of \( F \) are found to be positive in their respective regions of attraction, for each equilibrium point. Thus according to Theorem 5.6, the equilibrium points are globally stable in the positive orthants of their respective sub-domains.

Similarly a Liapunov function \( V(u, x_1, x_2, x_3) \) is constructed for the interior equilibrium \( E^* \) and further converted into the form \(-3F^T\), using equation 6.5.
The region of attraction for the system (6.9) is contained in the set A given as
\[ A = \{ (u, x_1, x_2, x_3) \mid 0 \leq u \leq 10, 0 \leq x_1 \leq 0.9, 0 \leq x_2 \leq 13.68, 0 \leq x_3 \leq 18.648 \} \]
The minimum values of the principal minors of F in the set A are given below.
\[ F_1 = 1, \ F_2 \geq 0.1658, \ F_3 \geq 0.002, \ F_4 \geq 0.0000156. \]

As the minimum values of principal minors of F are all positive in the region of attraction, according to Theorem 5.6 the interior equilibrium is globally stable in \( \mathbb{R}^4 \).

Computing the variational matrix for each boundary equilibrium point and finding their eigenvalues we find that all the boundary equilibria are hyperbolic saddle points in \( \mathbb{R}^4_{u,x_1,x_2,x_3} \). In this way all the conditions of Theorem 6.9 are satisfied and the system (6.9) is proved to be a uniformly persistent system.