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PUBLISHED PAPERS:


ACCEPTED PAPER:

Vibrational Analysis of Cracked Rotor in Viscous Medium

D. R. PARHI
A. K. BEHERA

Department of Mechanical Engineering, Regional Engineering College, Rourkela - 8, Sundargrah, Orissa 769008, India

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Abstract: This paper presents the vibrational analysis of a rotating cracked shaft in viscous medium. Virtual mass effect and damping effect on rotating shaft due to viscous fluid are analyzed with the help of Navier Stoke's equation. Analyses of the cracked portion of the shaft for fundamental frequency are done with the help of influence coefficients, stiffness elements, and boundary conditions at the cracked section. It is observed that, because of the crack and viscous fluid, there is a change in critical speed and amplitude of vibration.

Key Words: Crack depth, crack position, viscous medium, virtual mass effect, damping effect

NOMENCLATURE

\( A_1 \) = Shaft cross-sectional area
\( a_1 \) = Crack depth
\( b \) = Half the width of the crack
\( D \) = Diameter of the shaft
\( \delta \) = Whirling radius of the shaft
\( E \) = Modulus of elasticity of shaft material
\( e \) = Eccentricity
\( e_1 \) = Eccentricity in \( 44 \)-direction
\( e_2 \) = Eccentricity in \( 55 \)-direction
\( F_x, F_y \) = Fluid forces on rotor in \( x \) and \( y \) direction, respectively.
\( G \) = Shear modulus
\( I \) = Section moment of inertia of the shaft
\( \text{Im}[\cdot] \) = Imaginary part of [\cdot]
\( I_n(x) \) = Modified Bessel function of first kind of order \( n \)
\( L \) = Total length of the shaft
\( L_1 \) = Crack position from left side of the shaft
\( k \) = \( \sqrt{i\omega/\nu} \), where \( i = \sqrt{-1} \)
\( K_{55}, K_{44} \) = Stiffnesses of the cracked shaft in two directions (55- and 44-)
\( kk \) = \( 6(1 + v_i)/(7 + 6v_i) \)
\( [K_k] \) = Global stiffness matrix
\( K_n(x) \) = Modified Bessel function of second kind of order \( n \)
\( L \) = Length of span
\( L_x \) = Crack position from left of the span
\( m_s \) = Mass of the shaft per unit length
\( m \) = Fluid mass displaced by the shaft per unit length
\( \rho \) = Pressure
\( R_1 \) = Radius of the shaft
\( R_2 \) = Radius of the cylinder
\( t \) = Time
\( u \) = Radial flow velocity
\( v \) = Tangential flow velocity
\( a \) = \( kR_1 \)
\( a_a \) = Relative crack position \((L_x/L)\)
\( \beta \) = \( kR_2 \)
\( \beta_b \) = Relative crack depth \((a_1/D)\)
\( \mu \) = Coefficient of viscosity
\( \nu \) = Coefficient of kinematic viscosity
\( \nu_1 \) = Poisson's ratio
\( \rho \) = Fluid density
\( \rho_1 \) = Mass density of the shaft
\( \psi, \psi_1, \psi_2 \) = Stream functions
\( \omega \) = Rotating speed
\( \omega_0 \) = Natural angular frequency of the uncracked rotor in air
\( \omega_{xx}, \omega_{yy} \) = Critical speeds in \( x \) and \( y \) directions, respectively
\( \Omega \) = Angular velocity of whirling
\( 44\text{-Dir}^n \) = Direction perpendicular to crack
\( 55\text{-Dir}^n \) = Direction along the crack

1. INTRODUCTION

For decades, engineers have concentrated on the critical speed of rotors in vibrational analysis, under various conditions, because of its importance in design. When a cracked shaft rotates in viscous medium, the analysis of critical speed becomes complex.


Walston, Ames, and Clark (1964) in the their paper discussed the behavior of a rotating shaft in viscous fluid, but no clear-cut distinction was made between virtual mass effect and damping effect on the rotating shaft.

Parhi, Behera, and Behera (1995) in their paper discussed the vibrational behavior of a cracked cantilever beam with the help of influence coefficients at the crack section. They also verified their results experimentally. Mayes and Davies (1976) described a theoretical and experimental project designed to locate a transverse crack in the rotor systems. Also, Nelson and Nataraj (1986) discussed the same problem using perturbation technique. Sol (1980)
proposed an analytical model and correlated his results with those from an experimental rig. He demonstrated that the presence of a crack influences vibration amplitudes at the critical speed. Papadopoulos and Dimarogonas (1987) discussed the behavior of a rotating cracked shaft, but virtual mass effect (added mass) and external damping effect were not discussed separately.

Crighton (1983) analyzed the resonant oscillations of fluid-loaded struts in his paper. Achenbach and Qu (1986) in their paper discussed the resonant vibrations of a submerged beam of circular cross-section using mathematical methods. They compared the exact results for the beam deflection with the approximate results that are available in the literature. Gasch (1993) in his paper analyzed the dynamic behavior of a simple rotating shaft with a transverse crack. He took the De-Laval rotor for his analysis and explored some possibilities for early crack detection. The investigations carried out in the above papers give some way to tackle the vibration analysis problem of a rotor in viscous fluid.

In this paper, a systematic analysis for the vibrational behavior of a rotating cracked shaft in viscous medium is presented. Virtual mass effect and damping effect due to viscous fluid are determined with the help of Navier Stoke’s equation. Natural frequency of the cracked shaft, used for finding the critical speed of the system, is determined using the influence coefficients, stiffness elements, and boundary conditions at the cracked section.

The analysis of a cracked rotating shaft in viscous medium will be utilized for condition monitoring in various machineries; for early crack detection in rotor; for vibration analysis of (a) high-speed rotor in centrifuges prone to fatigue, (b) high-speed boring machines, and (c) rotors used for drilling oil from the sea bed; and for preventing failure of rotors used in machineries subjected to various environmental conditions.

2. ANALYSIS OF FLUID MOTION

The cracked shaft of radius $R_1$ is rotating with a speed $\omega$ having whirling speed $\Omega$ with $\delta$ as the radius, as shown in Figure 1.

The Navier-Stokes equation in polar coordinate can be expressed as

$$\begin{align*}
\frac{\partial u}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + v \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - u \frac{\partial}{\partial r} - \frac{u}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{2 \partial v}{r^2} \frac{\partial}{\partial \theta} \right), \\
\frac{\partial v}{\partial t} &= -\frac{1}{r \rho} \frac{\partial p}{\partial \theta} + v \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{2 \partial u}{r^2} \frac{\partial}{\partial \theta} \right).
\end{align*}$$

In the above equation, $u$ and $v$ denote the flow velocity in radial and tangential direction, respectively, and $p$ is the fluid pressure. With the help of stream function $\psi(r, \theta, t)$, the above equation can be written as

$$\nabla^4 \psi - \frac{1}{\nu} \frac{\partial}{\partial t} \nabla^2 \psi = 0,$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$.

Equation (2) can be divided into two parts, that is,

$$\nabla^2 \psi = 0, \quad \nabla^2 \psi - \left( \frac{1}{\nu} \frac{\partial}{\partial t} \right) = 0.$$
The solution of equation (2) can be given by

\[ \psi = \psi_1 + \psi_2, \]

where \( \psi_1 \) and \( \psi_2 \) are solutions of equation (3).

The radial and tangential components of flow velocity at point \( A \) in Figure 1 are

\[ \begin{align*}
  u_a &= R_1 \omega \sin (a) - \delta \Omega \sin (\Omega t - \theta) \\
  v_a &= R_1 \omega \cos (a) + \delta \Omega \cos (\Omega t - \theta)
\end{align*} \]

where \( a \) is the angle between \( O'A \) and \( OA \).

\[ \sin (a) = (\delta / R_1) \sin (\Omega t - \theta) \quad \text{and} \quad \cos (a) = 1 \quad \text{for} \ \delta << R_1, \]

for \( r = R_1 \), equation (4) can be rewritten
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\[ u \big|_{r=R_1} = \delta (\omega - \Omega) \sin (\Omega t - \theta) = \text{Re} \left( -i \delta (\omega - \Omega) e^{i(\Omega t - \theta)} \right) \]

\[ v \big|_{r=R_1} = \delta \Omega \cos (\Omega t - \theta) + R_1 \omega = \text{Re} \left( \delta \Omega e^{i(\Omega t - \theta)} \right) + R_1 \omega. \]

where \( i = \sqrt{-1} \) and \( \text{Re} \left[ . \right] \) denotes the real part of \( . \). For special cases (i.e., synchronous whirl), \( \omega = \Omega = \text{Constant} \), equation (5) reduces to

\[ u \big|_{r=R_1} = 0, \quad v \big|_{r=R_1} = \text{Re} \left[ \delta \omega e^{i(\omega t - \theta)} \right] + R_1 \omega. \] (6)

When the shaft is immersed in an infinitely extending fluid region, the boundary conditions \( \{(r \to \infty)\} \), that is, the container radius, is taken as \( \infty \) are taken as

\[ u \big|_{r=\infty} = v \big|_{r=\infty} = 0. \] (7)

Under these conditions, nonstationary components of the solutions \( \psi_1 \) and \( \psi_2 \) can be expressed as

\[ \psi_1 (r, \theta, t) = F_1 (r) e^{i(\omega t - \theta)}, \quad \psi_2 (r, \theta, t) = F_2 (r) e^{i(\omega t - \theta)}. \] (8)

From equation (8) and equation (3), we obtain

\[ \frac{d^2 F_1}{dr^2} + \frac{1}{r} \left( \frac{dF_1}{dr} - \frac{1}{r^2} F_1 \right) = 0 \quad \text{(9a)} \]

\[ \frac{d^2 F_2}{dr^2} + \frac{1}{r} \left( \frac{dF_2}{dr} - \frac{1}{r^2} F_2 + \frac{1}{r^2} + k^2 \right) F_2 = 0, \quad \text{(9b)} \]

where \( k = \sqrt{i \omega / \nu} \).

Equation 9(a) is Euler's equation, and equation 9(b) is Bessel's equation. The solution to the above equations can be written as

\[ F_1 (r) = \delta \omega \left( AR_1 / r + Br \right), \quad F_2 (r) = \delta \omega R_1 (CI_1 (kr) + DK_1 (kr)) \] (10)

where \( A, B, C, \) and \( D \) are arbitrary constants and \( I_1 (kr) \) and \( K_1 (kr) \) are modified Bessel's function of the first and second kind, respectively. The nonstationary component of flow velocity can be written as

\[ u_i = -\frac{1}{r} \frac{\delta \psi}{\delta \theta} = i \delta \omega \left( A \left( \frac{R_1}{r} \right)^2 + B + C I_1 (kr) + D K_1 (kr) \right) e^{i(\omega t - \theta)} \]

\[ v_i = \frac{\delta \psi}{\delta r} = \delta \omega \left[ -A \left( \frac{R_1}{r} \right)^2 + B + C \left\{ - \frac{R_1}{r} I_1 (kr) + k R_1 I_0 (kr) \right\} \right. \]

\[ + D \left\{ - \frac{R_1}{r} K_1 (kr) - k R_1 K_0 (kr) \right\} \right] e^{i(\omega t - \theta)} \] (11)
3. ANALYSIS OF FLUID FORCES

Substituting the flow velocities given by equation (11) to equation (1), the nonstationary component of pressure $p$ can be written as

$$p = \int \frac{\partial p}{\partial \theta} d\theta = \delta \rho \omega^2 \left\{ -\frac{A}{r} R_1^2 + Br \right\} e^{i(o\tau - \theta)}. \quad (12)$$

Normal stress $\tau_{rr}$ and tangential stress $\tau_{r\theta}$ due to flow can be obtained as

$$\tau_{rr} = -p + 2\mu \frac{\partial u}{\partial r} \quad \text{and} \quad \tau_{r\theta} = \mu \left( \frac{r}{\partial r} \left( \frac{\partial u}{\partial r} \right) + \frac{1}{r} \frac{\partial u}{\partial \theta} \right). \quad (13)$$

Fluid forces acting on the surface (i.e., $r = R$) per unit length of the shaft in the $x$ and $y$ direction are obtained by

$$F_x = \int_0^{2\pi} (\tau_{rr} \cos \theta - \tau_{r\theta} \sin \theta) R_1 d\theta = m\delta \omega^2 \{ A - B - CI_1(\alpha) - DK_1(\alpha) \} e^{i\omega t}$$

$$F_y = \int_0^{2\pi} (\tau_{rr} \sin \theta + \tau_{r\theta} \cos \theta) R_1 d\theta = -im\delta \omega^2 \{ A - B - CI_1(\alpha) - DK_1(\alpha) \} e^{i\omega t}$$

where $\alpha = kR_1$, $m = \rho \pi R_1^2$.

Only the real parts of equation (14) are meaningful. So $F_x$ and $F_y$ after simplification can be expressed as

$$F_x = m\delta \omega^2 \left\{ \text{Re}(H) \cos (\omega t) - \text{Im}(H) \sin (\omega t) \right\} \quad (15)$$

$$F_y = m\delta \omega^2 \left\{ \text{Re}(H) \sin (\omega t) + \text{Im}(H) \cos (\omega t) \right\}$$

where $H = A - B - CI_1(\alpha) - DK_1(\alpha)$ and $\text{Re}(H)$, $\text{Im}(H)$ denote the real and imaginary part of $H$.

The coordinates of the center of the shaft (as shown in Figure 1) are $x = \delta \cos \omega t$, $y = \delta \sin \omega t$.

$$F_x = -m \text{Re}(H) \frac{d^2x}{dt^2} + m\omega \text{Im}(H) \frac{dx}{dt}$$

$$F_y = -m \text{Re}(H) \frac{d^2y}{dt^2} + m\omega \text{Im}(H) \frac{dy}{dt} \quad (16)$$

In equation (16), $m \text{Re}(H)$ denotes the virtual or added mass to the inertia force of the shaft and $-m\omega \text{Im}(H)$ denotes the viscous damping coefficient.

4. ANALYSIS OF ROTOR MOTION

Here, a simply supported rotating cracked shaft immersed in the fluid region is considered.

The equations of motion for the shaft having uniformly distributed mass and stiffness are
\[
\begin{align*}
m_s \frac{\partial^2 (x + \varepsilon \cos \omega t)}{\partial t^2} + EI \frac{\partial^4 x}{\partial x^4} &= F_x \\
m_s \frac{\partial^2 (x + \varepsilon \sin \omega t)}{\partial t^2} + E1 \frac{\partial^4 y}{\partial y^4} &= F_y
\end{align*}
\]

where \(m_s\) = mass of the shaft per unit length; 
\(m\) = fluid mass displaced by the shaft per unit length; and 
\(EI\) = bending stiffness of the shaft.

\[
\begin{align*}
m_s + m \text{Re}(H) & \frac{\partial^2 x}{\partial t^2} - m \text{Im}(H) \frac{\partial x}{\partial t} + E1 \frac{\partial^4 x}{\partial x^4} = m_s \varepsilon \omega^2 \cos \omega t \\
m_s + m \text{Re}(H) & \frac{\partial^2 y}{\partial t^2} - m \text{Im}(H) \frac{\partial y}{\partial t} + E1 \frac{\partial^4 y}{\partial y^4} = m_s \varepsilon \omega^2 \sin \omega t
\end{align*}
\]

Taking the eccentricity \(\varepsilon_1\) (perpendicular to the crack, i.e., along 44-Dir) and \(\varepsilon_2\) (along the crack, i.e., along 55-Dir), the analysis for amplitude is drawn. At first, \(\varepsilon_1\) is taken and its contribution in \(x\) and \(y\) direction is found out in the following procedure.

Introducing dimensionless quantities \(\xi = x/R_1\), \(\eta = y/R_1\), \(\zeta = z/R_1\), \(\varepsilon_1^* = \varepsilon_1/R_1\), 
\(L^* = L/R_1\), \(m^* = m/m_s\), \(\omega_1^* = \omega/\omega_1\), \(\omega_2^* = \omega/\omega_2\), \(f_1 = (\omega_0/\omega_1)^2\), \(f_2 = (\omega_0/\omega_2)^2\), 
\(\tau_1 = \omega_1 t\), \(\tau_2 = \omega_2 t\) \((\omega_1, \omega_2)\) are the fundamental natural frequencies in \(x\) and \(y\) directions, respectively, of the cracked shaft as shown in Figure 2. Whereas \(\omega_0\) is the fundamental natural frequency of the uncracked shaft \(t = \pi^2(EI/(m_s L^4))^{0.5}\), equation (18) reduces to

\[
\begin{align*}
\{1 + m^* \text{Re}(H)\} & \frac{\partial^2 \xi}{\partial \tau_1^2} - m^* \omega_1^* \text{Im}(H) \frac{\partial \xi}{\partial \tau_1} + f_1 \frac{\omega_1^*}{\pi^4} \frac{\partial^4 \xi}{\partial \xi^4} = \varepsilon_1^* (\omega_1^*)^2 \cos (\omega_1^* \tau_1) \\
\{1 + m^* \text{Re}(H)\} & \frac{\partial^2 \eta}{\partial \tau_2^2} - m^* \omega_2^* \text{Im}(H) \frac{\partial \eta}{\partial \tau_2} + f_2 \frac{\omega_2^*}{\pi^4} \frac{\partial^4 \eta}{\partial \eta^4} = \varepsilon_1^* (\omega_2^*)^2 \sin (\omega_2^* \tau_2)
\end{align*}
\]

Applying the Fourier transform to both sides of equation (19), we obtain

\[
\begin{align*}
X_\nu (\tau_1) + C_{\nu 1} \tilde{X}_\nu + K_{\nu 1} X_\nu &= A_{\nu 1} \cos (\omega_1^* \tau_1) \ A_{\nu 1} = \frac{1 + m^* \text{Re}(H)}{1 + m^* \text{Re}(H)} \\
Y_\nu (\tau_2) + C_{\nu 2} \tilde{Y}_\nu + K_{\nu 2} Y_\nu &= A_{\nu 2} \sin (\omega_2^* \tau_2) \ A_{\nu 2} = \frac{1 + m^* \text{Re}(H)}{1 + m^* \text{Re}(H)} \end{align*}
\]

The steady-state solution of equation (20) is easily obtained as \(X_\nu = (1/2) \varepsilon_1^* \omega_1^* \sin (\nu \pi/2)\), \(Y_\nu = (1/2) \varepsilon_1^* \omega_1^* \cos (\nu \pi/2)\) (for \(n = 1, 2, \ldots\)), where
Cross Sectional View of the shaft at the position of crack

Figure 2. Simply supported shaft with crack.

\[ \delta_{vn}^* = \frac{A_{vn}}{\sqrt{(K_{vn} - (\omega_n^*)^2 + (C_{vn} \omega_n^*)^2), \quad \phi_{vm} = \tan^{-1} \left( \frac{C_{vn} \omega_n^*}{K_{vn} - (\omega_n^*)^2} \right) \right.} \]

for \( v = 1, 2, \ldots; n = 1 \& 2 \)

Taking the inverse Fourier transform of \( X_v \), we get

\[ \zeta (\zeta, \tau_1) = 2 \sum_{v=1}^{\infty} X_v (\tau_1) \sin(v \pi \zeta / L^*). \]

Similarly, for \( Y_v \) it can be done.

From the above equations, the whirling motion for fundamental bending mode in \( x \) and \( y \)-Dir. can be written, respectively, as
\[ \begin{align*}
\zeta \left( \frac{L^*}{2}, \tau_1 \right) &= \delta_{11}^* \cos(\omega_1^* \tau_1 - \phi_{11}) \\
\eta \left( \frac{L^*}{2}, \tau_2 \right) &= \delta_{12}^* \sin(\omega_2^* \tau_2 - \phi_{12})
\end{align*} \]

(21)

\[ \zeta_{44} \left( \frac{L^*}{2}, \tau_1 \right) = \delta_{11}^* \cos(-\phi_{11}) \quad \text{in } x \text{ Direction} \]
\[ \eta_{44} \left( \frac{L^*}{2}, \tau_2 \right) = \delta_{12}^* \sin(\omega_2^* \tau_2 - \phi_{12}), \text{when } \omega_1^* \tau_1 = 0 \quad \text{in } y \text{ Direction} \]

(22)

Similarly, the expression for \( \zeta \left( \frac{L^*}{2}, \tau_1 \right) \) and \( \eta \left( \frac{L^*}{2}, \tau_1 \right) \) due to the eccentricity \( \varepsilon_2 \) (eccentricity in the Dir" of the crack, 55-Dir") can be found out by adopting the above procedure.

When the 55-Dir" axis coincides with the \( y \) axis, the amplitude contribution of \( \varepsilon_1 \) in the \( x \) and \( y \) directions is \( \zeta_{55} \left( \frac{L^*}{2}, \tau_2 \right) \) and \( \eta_{55} \left( \frac{L^*}{2}, \tau_2 \right) \), respectively.

The total dimensionless deflection in \( x \) and \( y \) directions, when the 44-Dir" (perpendicular to the crack) and 55-Dir" (along the crack) coincide with the \( x \) axis and \( y \) axis, respectively, is

\[ \begin{align*}
\delta_1^* &= \delta_{44}^* = \zeta_{44} \left( \frac{L^*}{2}, \tau_1 \right) + \zeta_{55} \left( \frac{L^*}{2}, \tau_1 \right) \quad \text{Along the } x \text{ (44-)Dir"} \\
\delta_2^* &= \delta_{55}^* = \eta_{44} \left( \frac{L^*}{2}, \tau_1 \right) + \eta_{55} \left( \frac{L^*}{2}, \tau_1 \right) \quad \text{Along the } y \text{ (55-)Dir"}
\end{align*} \]

(23)

\( \delta_1^* \) and \( \delta_2^* \) are the dimensionless amplitudes of the cracked rotor when 44-Dir" and 55-Dir" coincide with the \( x \) axis and \( y \) axis, respectively.

5. DISCUSSION

In Figure 1, a simply supported whirling cracked shaft having dimension (length = 1 m, radius = 0.008 m) is shown. In Figure 2, a simply supported cracked shaft is shown along with various coupling forces. In Figure 3 through Figure 6, comparisons for dimensionless amplitude ration in 44-Dir" (when the 44-Dir" coincides with the \( x \) axis) and in 55-Dir" (when the 55-Dir" coincides with the \( y \) axis) natural frequencies, for various viscous mediums, are shown.

For the above figures,

(i) the coefficients of viscosity (\( \nu \)) are 2.3, 0.427, 0.0633, and 0.0284 stokes.
(ii) the relative crack depths (\( \beta \)) are 0.2, 0.3, 0.4, and 0.5.
Figure 3. Frequency ratio \( \omega / \omega_0 \) versus dimensionless amplitude ratio \( (\delta_\alpha / \epsilon^* ) \), mild steel shaft specimen \( (R_1 = .008 \text{ m}, \ L = 1 \text{ m}) \), relative crack position \( (\alpha \alpha) = 0.5 \), relative crack depth \( (\beta \beta) = 0.2 \).

Figure 4. Frequency ratio \( \omega / \omega_0 \) versus dimensionless amplitude ratio \( (\delta_\alpha / \epsilon^* ) \), mild steel shaft specimen \( (R_1 = .008 \text{ m}, \ L = 1 \text{ m}) \), relative crack position \( (\alpha \alpha) = 0.5 \), relative crack depth \( (\beta \beta) = 0.3 \).
Figure 5. Frequency ratio ($\omega/\omega_0$) versus dimensionless amplitude ratio ($\delta^*/e^*$), mild steel shaft specimen ($R_i = 0.008$ m, $L = 1$ m), relative crack position ($\alpha\alpha$) = 0.5, relative crack depth ($\beta\beta$) = 0.4.

Figure 6. Frequency ratio ($\omega/\omega_0$) versus dimensionless amplitude ratio ($\delta^*/e^*$), mild steel shaft specimen ($R_i = 0.008$ m, $L = 1$ m), relative crack position ($\alpha\alpha$) = 0.5, relative crack depth ($\beta\beta$) = 0.5.
Figure 7. Frequency ratio ($\omega/\omega_0$) versus dimensionless amplitude ratio ($\delta_*/e*$), mild steel shaft specimen ($R_i = 0.008$ m, $L = 1$ m), relative crack position ($\alpha\alpha$) = 0.5, relative crack depth ($\beta\beta$) = 0.26.

Figure 8. Frequency ratio ($\omega/\omega_0$) versus dimensionless amplitude ratio ($\delta_*/e*$), mild steel shaft specimen ($R_i = 0.008$ m, $L = 1$ m), relative crack position ($\alpha\alpha$) = 0.5, relative crack depth ($\beta\beta$) = 0.36.
Figure 9. Damping effect.

Figure 10. Virtual mass effect.
(iii) the relative crack position \( (\alpha_1 \alpha_2) = 0.5 \).
(iv) \( e_1^* = e_2^* = e^* \)

It is noticed that, as the viscosity and density of the fluid increase, the amplitude as well as critical speed decrease.

From Figure 7 and Figure 8, it is observed that due to increase in crack depth, the critical speed decreases. The relative crack depths \( (\beta_1 \beta_2) \) taken for the two figures are 0.26 and 0.36, having relative crack position \( (\alpha_1 \alpha_2) = 0.5 \) and coefficient of viscosity \( (v) = 0.427 \) stokes.

It is noticed from the above analysis that the rotor passes through the resonant condition when the rotating speed \( (\omega) \) is between the 44-Dir\(^{\circ}\) and the 55-Dir\(^{\circ}\) natural frequency values. When the crack is in the opposite side of eccentricity (unbalance), the crack is in closed condition, for which the crack cannot propagate and is stable due to damping. But when it is in the same side, the crack is in opening condition. This will propagate due to centrifugal force, which will lead the amplitude of vibration to an unstable position.

In Figure 9, graphs are shown when the rotor is rotating in various damping mediums having unique virtual mass effect. It is observed that as the damping effect increases, the amplitude of vibration decreases. It is also noticed that damping effect has very little effect on resonance speed of the rotor.

From Figure 10, it is evident that as the virtual mass effect increases, the resonance speed decreases remarkably, as compared to amplitude of vibration.

To verify whether the virtual mass effect and viscous damping effect are authentic as they are incorporated in the theory developed, the theoretical results (Appendix B) are compared with the experimental results by Walston, Ames, and Clark (1964).
For experimental verification, the dimensions taken for cantilever type rotor are:
- Length of shaft = 0.762 m,
- Radius of shaft = 0.00635 m,
- Weight of the disk = 1.3 Kg,
- Radius of the disk = 0.054 m,
- Liquid medium = 70%, and glycerin = 100%.

It is observed that there is a very good agreement between the theoretical and experimental results.

6. CONCLUSIONS

From the above theoretical analysis and experimental verifications, the following conclusions may be depicted:

It is evident from Figure 3 through Figure 6 that, as the viscosity of the fluid increases, the critical speed decreases (due to increase in critical mass $mRe(H)$) and also the amplitude decreases (due to increasing in damping factor $-mIm(H)$).

From Figure 7 and Figure 8, it is concluded that, due to the presence of the crack, the critical speed decreases and, as the stiffness of the cracked shaft in the 55-direction is lower than the stiffness in the 44-direction, the critical speed in the 55-direction is lower than that of the 44-direction.

Because of the crack, the critical speed of the shaft decreases, and due to that low critical speed, the damping coefficient $-mIm(H)$ increases for which the maximum dimensionless amplitude of the rotating cracked shaft is lower than that of the uncracked shaft. The above-mentioned effect can be visualized from Figure 7 and Figure 8.

External damping has got more impact in reducing the amplitude of vibration than in changing the resonance speed. This effect can be visualized from Figure 9. But virtual mass effect reduces the resonance speed considerably more than it reduces the amplitude of vibration.

The above-mentioned effect may be utilized for analysis of a cracked rotating shaft in viscous medium; for condition monitoring in machines; for early crack detection in a rotor; for vibration analysis of (a) high-speed rotor in centrifuges prone to fatigue, (b) high-speed boring machines, and (c) rotors used for drilling oil from sea bed; and for preventing failure of rotors used in machineries subjected to various environmental conditions.

APPENDIX A

ANALYSIS OF THE VIBRATION CHARACTERISTIC OF THE CRACKED SHAFT

A simply supported shaft of length $L$ radius $R_1$, with a crack of depth $a_1$ at a distance $L_1$ from one end is being considered (as shown in Figure 2). Taking $\Theta_1(z, t)$ and $\Theta_2(z, t)$ as deflection of torsional vibrations for the sections before and after the crack, $Y_1(z, t)$ and $Y_2(z, t)$ are the deflection for bending vibrations and $\xi_1(z, t)$, $\xi_2(z, t)$ are the slope of the deflection curve for the same sections. The normal functions of vibration, Thomson (1988), for the system in dimensionless form can be defined as...
\[ 
\begin{align*}
\bar{Y}_1(z) &= A_1 \cos(\bar{K}_u \bar{z}) + A_2 \sin(\bar{K}_u \bar{z}) \\
\bar{Y}_2(z) &= A_3 \cos(\bar{K}_u \bar{z}) + A_4 \sin(\bar{K}_u \bar{z})
\end{align*}
\]

where
\[ \bar{z} = \frac{z}{L}, \quad \bar{y}_i = \frac{Y_i}{L}, \quad \bar{\xi}_i = \frac{\xi_i}{L} \quad \text{for} \ i = 1, 2 \]

\[ \bar{K}_u = \frac{e_0 L}{C_w}, \quad \mu \mu = A_1 \rho_1 \]

\[ \lambda^2 = \frac{\mu \mu L^2 \omega^2}{A_1 E} \left( 1 + \frac{E}{\kappa G} \right), \quad \psi \psi^2 = \frac{\mu \mu L^4 \omega^2}{EI} \left( 1 + \frac{\rho_1 I \omega^2}{\kappa G A_1} \right) \]

\[ \lambda_1 = \left( \sqrt{\psi \psi^2 + \lambda_2^2/4 + \lambda_3^2/2} \right)^{0.5}, \quad \lambda_2 = \left( \sqrt{\psi \psi^2 + \lambda_2^2/4 - \lambda_3^2/2} \right)^{0.5} \]

\[ A_1 = (\lambda_1 - \lambda_3)/\lambda_1, \quad A_2 = (\lambda_2 + \lambda_3)/\lambda_2, \quad \lambda_3 = \omega/\left[ \kappa G A_1 / (\mu \mu L^2) \right]^{0.5} \]

\[ A_{i1}, A_{i2}, A_{i3}, A_{i4}, \rho_1, b_1, c_1, d_1, a_2, b_2, c_2, \text{ and } d_2 \] are the constants to be determined from the boundary conditions.

The boundary conditions of the simply supported shaft in consideration are

\[ Y_1(0) = 0 \]

\[ E I \xi^2_1(0) = 0 \]

\[ Y_1(L) = 0 \]

\[ E I \xi^2_2(0) = 0 \]

\[ E I \xi^2_1(L_1) = E I \xi^2_2(L_1) \]

\[ \kappa G A_1 (Y_1'(L_1) - \xi_1(L_1)) = \kappa G A_1 (Y_2'(L_1) - \xi_2(L_1)) \]

\[ \kappa G A_1 (Y_1''(L_1) - \xi_1''(L_1)) = \kappa G A_1 (Y_2''(L_1) - \xi_2''(L_1)) \]

\[ G L_1 \xi_1'(0) = 0 \]

\[ G L_1 \xi_1'(L_1) = 0 \]

\[ G L_1 \xi_2'(L_1) = G L_1 \xi_2'(L_1) \]

\[ G L_1 \xi_1'(L_1) = G L_1 \xi_2'(L_1) \]
Equations (2) to (13) are for 44-Dir 

\( KK_{ij} \) = local flexibility matrix elements are obtained by taking the inversion of the compliance matrix.

Compliance matrix elements \( C_{ij} \) (influence coefficients) can be obtained by taking the strain energy (see Tada, Paris, and Irwin, 1973), at crack location.

\[
C_{ij} = \frac{a^2}{\partial P_i \partial P_j} \int_{-b}^{a} \int_{a_2}^{a_3} J(a) \, dy \, dx.
\]

\( J(a) \) = strain energy density function; \( P_i, i = 1 \) to 6.

Axial force (\( i = 1 \))

Shear force (\( i = 2, 3 \))

Bending moment (\( i = 4, 5 \))

Torsional moment (\( i = 6 \))

\( 2b \) = width of the crack, \( a_1 \) = depth of the crack; \( a_2 = R_1 - a_1, a_3 = h1/2 \) (\( h_1, a_1, a_2, a_3 \) are shown in Figure 2).

Applying boundary conditions in characteristic equation, the fundamental critical speed in 55-(\( \omega_{55} \)) and 44-(\( \omega_{44} \)) is found out. The global stiffness matrix \( [K_g] \) can be written as

\[
[K_g] = \begin{bmatrix}
K_{55} & K_{54} & K_{51} \\
K_{45} & K_{44} & K_{41} \\
K_{15} & K_{14} & K_{11}
\end{bmatrix} = \left( [G_1] [C_{\sigma \tau}] [G_2] + [C_\sigma]\right)^{-1},
\]

where \( [G_1] = \text{diag} \left[ L/2, L/2, 1 \right], [G_2] = \left[ L/4, L/4, 1 \right], \)

\[
[C_\sigma] = \text{diag} \left[ \frac{L^3}{48E\ell^2}, \frac{L^3}{48E\ell^2}, \frac{L}{EA_1} \right], [C_{\sigma \tau}] = \frac{1}{F_0} \begin{bmatrix}
\bar{C}_{55}/R_1 & \bar{C}_{54}/R_1 & \bar{C}_{51} \\
\bar{C}_{45}/R_1 & \bar{C}_{44}/R_1 & \bar{C}_{41} \\
\bar{C}_{15} & \bar{C}_{14} & \bar{C}_{11}/R_1
\end{bmatrix}
\]

where \( F_0 = A1E/(1 - \nu^2) \) and \( \text{diag} \left[ \ldots \right] = \text{diagonal matrix}. \)

**APPENDIX B**

Here, the analysis of a cantilever-type rotor having a disk at the end span is done. An equivalent lumped mass of a rotating shaft is given by \( K_r/\omega_1^2 \), where \( K_r \) and \( \omega_1 \) are the stiffness and fundamental natural frequency shaft, respectively.

The ratio of the equivalent lumped mass to the total mass of the shaft is given by the expression

\[
\alpha_{eq} = \frac{K_r}{\omega_1^2 M_{12}},
\]

where \( M_{12} \) is the mass of the shaft.

If a disk with mass \( M_{11} \) is attached to the end span of the shaft, a total lumped mass of the rotor becomes

\[
M_r = M_{11} + \alpha_{eq} M_{12}.
\]
The equation of motion of an equivalent single degree of freedom system of the whirling cracked rotor in fluid is reduced to

\[ M_s \frac{d^2}{dt^2} (x + \varepsilon \cos \omega t) + K_s x = F_x \]

\[ M_s \frac{d^2}{dt^2} (y + \varepsilon \sin \omega t) + K_s y = F_y \]

Equation (B1)

The fluid forces from equation (16) can be written as

\[ F_x = -M \text{Re}(H) \frac{d^2 x}{dt^2} + M_0 \text{Im}(H) \frac{dx}{dt} \]

\[ F_y = -M \text{Re}(H) \frac{d^2 y}{dt^2} + M_0 \text{Im}(H) \frac{dy}{dt} \]

Equation (B2)

where

\[ M \text{Re}(H) = M_1 \text{Re}(H_1) + \alpha_{eq} M_2 \text{Re}(H_2) \]

\[ M \text{Im}(H) = M_1 \text{Im}(H_1) + \alpha_{eq} M_2 \text{Im}(H_2) \]

\[ M = M_1 + \alpha_{eq} M_2 \]

where \( M_1 \) and \( M_2 \) mass of the fluid displaced by the disk and shaft respectively.

From equations (B1) and (B2), we have

\[ (M_s + M \text{Re}(H)) \frac{d^2 x}{dt^2} - M_0 \text{Im}(H) \frac{dx}{dt} + K_s x = M_s \varepsilon \omega^2 \cos \omega t \]

\[ (M_s + M \text{Re}(H)) \frac{d^2 y}{dt^2} - M_0 \text{Im}(H) \frac{dy}{dt} + K_s y = M_s \varepsilon \omega^2 \sin \omega t \]

Equation (B3)

Equation (B3) in dimensionless form can be written as

\[ \left\{ 1 + M^* \text{Re}(H) \right\} \frac{d^2 \xi}{d\tau^2} - M^* \omega^* \text{Im}(H) \frac{d\xi}{d\tau} + \xi = \varepsilon^* (\omega^*)^2 \cos (\omega^* \tau) \]

\[ \left\{ 1 + M^* \text{Re}(H) \right\} \frac{d^2 \eta}{d\tau^2} - M^* \omega^* \text{Im}(H) \frac{d\eta}{d\tau} + \eta = \varepsilon^* (\omega^*)^2 \sin (\omega^* \tau) \]

Equation (B4)

where \( \xi = x/R_1, \eta = y/R_1, \omega^* = \omega/\omega_{nr}, \varepsilon^* = \varepsilon/R, M^* = M/M_s, \tau = \omega_{nr} t, \omega_{nr} = \sqrt{K_s/M_s} \).

The steady-state solution of the above equation can be obtained in dimensionless form as

\[ \xi = \delta^* \cos (\omega^* \tau - \phi), \text{ where } \delta^* = \delta/R_1 \]

Equation (B5)
\[ \delta^* = \frac{A}{\sqrt{(K - (\alpha^*)^2)^2 + (\omega^*)^2}}, \quad \phi = \tan^{-1} \left( \frac{\omega^*}{K - (\omega^*)^2} \right) \quad \text{and} \quad C = \frac{-M^*\alpha^*\text{Im}(H)}{1 + M^*\text{Re}(H)}, \quad K = \frac{1}{1 + M^*\text{Re}(H)}, \quad A = \frac{\varepsilon^* (\omega^*)^2}{1 + M^*\text{Re}(H)} \]  

(B6)

\( \delta^* \) is the maximum dimensionless amplitude.

REFERENCES


Vibration analysis of submerged cantilever type rotor system

Dayal R. Parhi* and A.K. Behera*

Forced vibration of a cantilever type submerged rotor is investigated in this paper. At first, from Navier Stokes equation the fluid forces acting on the rotor are found out. From the force analysis the virtual mass effect and damping effects on the rotor due to viscous fluid are calculated. Taking the disk (connected to end of the rotor) as lumped mass, the equation for amplitude of vibration of the system in viscous fluid of finite region is found out. Effects of viscosity, damping and virtual mass are shown by plotting the graphs.

Natural frequencies and mode shapes are important dynamic characteristics of a structure as they are required in the solution of resonant responses and forced vibration analysis. Practical applications of spinning structures can be found in rotating machinery systems, space structures, such as satellites, etc. But when the above cited structures are submerged under viscous medium, the vibration characteristics, i.e., amplitudes of vibration and natural frequencies, are affected greatly.

The study of free vibration for a stationary Timoshenko beam can be traced back to the middle of this century when Kruszewski1 obtained the frequency equations for the fixed-free and the free-free beams. Anderson2 and Dolph3 presented general solutions for the simply-supported boundary conditions. A complete theoretical treatment of the problem was published by Trail-Nash and Collar4, giving the frequency equations and mode shapes for all six types of boundary condition, namely, simply-supported or hinged-hinged, hinged-free, free-free, fixed-fixed, fixed-free, and fixed-hinged. However, they only presented results for half the mode shapes as those associated with the bending were not derived. Using a more systematic approach, Huang5 provided complete results, both with regard to the frequency equations and the mode shapes. With the exception of the simply-supported beam, the solution of the frequency equations requires a fair amount of computational effort as they are transcendental in nature. This led to the popularity of approximate techniques, mainly based on energy methods, for analysis and solution. Huang6, and later Hurty and Rubenstein7, gave some results for the simply-supported beam. Carr8 presented solutions for all the six boundary conditions.

The study of spinning finite beams was treated in a comprehensive manner by Bauer9, where he derived the relationship between natural frequencies of a spinning and a stationary beam, based on the Euler beam theory. His analysis is valid for any of the classical boundary conditions. Lee et al.10 obtained closed-form solutions for frequencies and mode shapes, using the Rayleigh beam model, for only two types of boundary conditions. For the Timoshenko beam model, a thorough investigation, which is valid for simply-supported beams, was carried out by Dimentberg11. Using an integral transform technique, Katz et al.12 developed the frequency equation, also for simply-supported beams. A different approach, one based on the physically more meaningful modal analysis, was presented by Han and Zu13 in their solution for the frequency response of a simply supported Timoshenko beam.

From the above survey, it is noticed that the resonant frequency is the governing factor in the analysis of vibration characteristics. This effect has got enormous impact on vibration characteristics, if the vibrations in rotors are taking place in submerged (inside the fluid) state.

The vibration motion of a submerged solid body is greatly affected by the viscosity of the fluid. The resonant frequencies differ from the ones in vacuum. The effect along with damping and virtual mass are analysed quantitatively by the use of mathematical methods in the present investigation. The particular system being taken here,
concerns with the forced transverse vibrations of a cantilever type submerged rotor having a disk at the end span.

ANALYSIS OF FLUID MOTION

A shaft of radius \( R \), rotating with a speed \( \omega \), having whirling speed \( \Omega \) with \( i \) as the radius, is shown in Fig. 1b.

The Navier-Stokes equation in polar co-ordinates is generally expressed as:

\[
\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + v \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right)
\]

\[
\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial \theta} + v \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{u}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right)
\]

In the above equations \( u \) and \( v \) denote the flow velocity in radial and tangential direction respectively. With the help of stream function \( \psi (r, \theta, t) \) the above equations can be written as:

\[
\nabla^4 \psi = -\frac{1}{v} \frac{\partial}{\partial t} \nabla^2 \psi = 0
\]

where, \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \)

Equation (2) can be divided into two parts, i.e.

\[
\nabla^2 \psi = 0, \quad \nabla^2 \psi - \left( \frac{1}{v} \frac{\partial \psi}{\partial t} \right) = 0
\]

The solution of Eq. (2) is expressed as:

\[
\psi = \psi_1 + \psi_2
\]

The radial and tangential components of flow velocity at point \( A \) in Fig. 1 are:

\[
u_r = R_1 \omega \sin (\alpha) - \delta \Omega \sin (\Omega t - \theta)
\]

\[
u_t = R_1 \omega \cos (\alpha) + \delta \Omega \cos (\Omega t - \theta)
\]

where \( \alpha \) is the angle between \( OA' \) and \( OA \)

\[
\sin (\alpha) = \left( \frac{R}{R_1} \right) \sin (\Omega t - \theta) \quad \text{&} \quad \cos (\alpha) = 1 \text{ for } \delta < R_1
\]

For \( r = R_1 \), Eq. (4) can be rewritten as:

\[
u_r = R_1 = \delta (\omega - \Omega) \sin (\Omega t - \theta) = \text{Re} \left[ i \delta (\omega - \Omega) e^{i(\Omega t - \theta)} \right]
\]

\[
u_t = R_1 \omega \cos (\Omega t - \theta) + R_1 \omega = \text{Re} \left[ \delta \Omega e^{i(\Omega t - \theta)} \right] + R_1 \omega
\]

(5)

Where, \( i = \sqrt{-1} \) and \( \text{Re} [\cdot] \) denotes the real part, Taking \( \omega = \Omega \), Eq. (5) reduce to;

\[
u_r = R_1 = 0, \quad \nu_t = R_1 = \text{Re} \left[ \delta \omega e^{i(\omega t - \theta)} \right] + R_1 \omega
\]

(6)

When the shaft is immersed in finite extending fluid region, the boundary conditions \(( r = R_2 \), i.e., the container radius is taken as \( R_2 \) \) are:

\[
u_r = R_2 = 0, \quad v_t = R_2 = 0
\]

(7)

Under these conditions, non-stationary components of the solutions \( \psi_1 \) and \( \psi_2 \) can be expressed as:

\[
\psi_1 (r, \theta, t) = F_1 (r) e^{i(\omega t - \theta)}
\]

\[
\psi_2 (r, \theta, t) = F_2 (r) e^{i(\omega t - \theta)}
\]

(8)

From Eq. (8) and Eq. (3), we obtain;

\[
\frac{d^2 F_1}{dr^2} + \frac{1}{r} \frac{dF_1}{dr} - \frac{1}{r^2} F_1 = 0 \quad \text{(9.a)}
\]

\[
\frac{d^2 F_2}{dr^2} + \frac{1}{r} \frac{dF_2}{dr} \left( \frac{1}{r^2} + k^2 \right) F_2 = 0 \quad \text{(9.b)}
\]

where \( k = \sqrt{\frac{i \omega}{v}} \)

Eq. (9.a) is Euler's equation and (9.b) is Bessel's equation. The solution to the above equations can be written as;

\[
F_1 (r) = \delta \omega (AR^2/r + Br)
\]

\[
F_2 (r) = \delta \omega R_1 \left[ C \psi_1 (kr) + DK \psi_2 (kr) \right]
\]

(10)

where \( A, B, C \) and \( D \) are arbitrary constants and \( \psi_1 (kr) \) and \( K \psi_2 (kr) \) are modified Bessel's function of 1st. and 2nd. kind respectively. The non-stationary components of flow velocities can be written as;

\[
u_r = \frac{\partial \psi}{\partial \theta} = \delta \omega \left[ \frac{R^2}{r^2} + B + C \psi_1 (kr) + D K \psi_2 (kr) \right] e^{i(\omega t - \theta)}
\]

(11)
ANALYSIS OF FLUID FORCES

Substituting the flow velocities given by Eq. (11) into Eq. (1), the non-stationary component of pressure \( p \) can be written as:

\[
p = \int \frac{\partial p}{\partial t} \, dt = \delta \rho \omega^2 \left( -\frac{A}{r^3} + B r \right) e^{i(\omega t - \theta)}
\]

(12)

Normal and tangential stresses due to flow can be written as:

\[
\tau_{rr} = -p + 2\mu \frac{\partial u}{\partial r} \quad \text{and} \quad \tau_{r\theta} = \mu \left[ \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right]
\]

(13)

Fluid forces acting on the surface (i.e. \( r = R_1 \)) per unit length of the shaft in the x and y directions are written as:

\[
F_x = m\delta \omega^2 \left\{ \text{Re}(H) \cos(\omega t) - \text{Im}(H) \sin(\omega t) \right\}
\]

\[
F_y = m\delta \omega^2 \left\{ \text{Re}(H) \sin(\omega t) + \text{Im}(H) \cos(\omega t) \right\}
\]

(14)

Where, \( \alpha = kR_1 \), \( m = \rho \pi R_1^2 \)

Only the real parts of Eq. (14) has the meaning. So, \( F_x \) and \( F_y \) after simplification can be expressed as:

\[
F_x = -M\text{Re}(H) \frac{d^2x}{dt^2} + M\omega \text{Im}(H) \frac{dx}{dt}
\]

\[
F_y = -M\text{Re}(H) \frac{d^2y}{dt^2} + M\omega \text{Im}(H) \frac{dy}{dt}
\]

(15)

where, \( H = A - B - C \frac{1}{\omega} - D \frac{1}{\omega^2} \) and \( \text{Re}(H) \) and \( \text{Im}(H) \) denote the real and imaginary parts of \( H \).

The coordinates of the center of the shaft (Fig. 1b.) are:

\[
x = \delta \cos \omega t, \quad y = \delta \sin \omega t
\]

\[
F_x = -m\text{Re}(H) \frac{d^2x}{dt^2} + m\omega \text{Im}(H) \frac{dx}{dt}
\]

\[
F_y = -m\text{Re}(H) \frac{d^2y}{dt^2} + m\omega \text{Im}(H) \frac{dy}{dt}
\]

(16)

In Eq. (16), \( m\text{Re}(H) \) denotes the virtual or added mass of the fluid relating to the inertia force of the shaft and \( -m\omega \text{Im}(H) \) denotes the viscous damping co-efficient.

ANALYSIS FOR ROTOR AMPLITUDE

In the present analysis, a lumped mass at the end span of the cantilever type rotating shaft immersed in the finite fluid region is considered.

An equivalent lumped mass of a rotating shaft is given by \( K_s/\omega_i^2 \), where \( K_s \) and \( \omega_i \) are the stiffness and fundamental natural frequency of the shaft, respectively.

The ratio of the equivalent lumped mass to the total mass of the shaft is given by:

\[
\alpha_{eq} = \frac{K_s}{\omega_i^2 M_s}
\]

(17)

If a disk with mass \( M_{s1} \) is attached at the end span of the shaft, the total lumped mass of the rotor becomes:

\[
M_s = M_{s1} + \alpha_{eq} M_s
\]

The equation of motion of an equivalent single degree of freedom system of the whirling cracked rotor in fluid is reduced to:

\[
M_s \frac{d^2x}{dt^2} + K_s x = F_x
\]

\[
M_s \frac{d^2y}{dt^2} + K_s y = F_y
\]

The fluid forces from Eq. (16) can be written as:

\[
F_x = -M\text{Re}(H) \frac{d^2x}{dt^2} + M\omega \text{Im}(H) \frac{dx}{dt}
\]

\[
F_y = -M\text{Re}(H) \frac{d^2y}{dt^2} + M\omega \text{Im}(H) \frac{dy}{dt}
\]

(18)

where \( M \text{Re}(H) = M_1 \text{Re}(H_1) + \alpha_{eq} M_2 \text{Re}(H_2) \)

\( M \text{Im}(H) = M_1 \text{Im}(H_1) + \alpha_{eq} M_2 \text{Im}(H_2) \)

\( M = M_1 + \alpha_{eq} M_2 \), where \( M_1 \) and \( M_2 \) are mass of the fluid displaced by the disk and shaft respectively.

From Eqs. (17) and (18), we have:

\[
\begin{align*}
M_s \frac{d^2x}{dt^2} + K_s x & = M_1 \text{Re}(H_1) \frac{dx}{dt} + \alpha_{eq} M_2 \text{Re}(H_2) \frac{dx}{dt} \\
M_s \frac{d^2y}{dt^2} + K_s y & = M_1 \text{Im}(H_1) \frac{dy}{dt} + \alpha_{eq} M_2 \text{Im}(H_2) \frac{dy}{dt}
\end{align*}
\]

(19)

Eq. (19) in dimensionless form can be written as:

\[
\begin{align*}
\left[ M_s + M \text{Re}(H) \right] \frac{d^2x}{dt^2} + K_s x & = M_1 \text{Re}(H_1) \frac{dx}{dt} + \alpha_{eq} M_2 \text{Re}(H_2) \frac{dx}{dt} \\
\left[ M_s + M \text{Re}(H) \right] \frac{d^2y}{dt^2} + K_s y & = M_1 \text{Im}(H_1) \frac{dy}{dt} + \alpha_{eq} M_2 \text{Im}(H_2) \frac{dy}{dt}
\end{align*}
\]

(20)

where, \( \xi = x/R_1, \eta = y/R_1, \omega_s = \omega/\omega_n, \epsilon = t/\tau \)

\( M* = M/M_s, \quad \tau = \omega_n t, \quad \omega_n = \sqrt{K_s/M_s} \)

The steady state solution of the above equation can be obtained in dimensionless form as:

\[
\xi = \delta^* \cos(\omega^* t - \phi), \quad \text{Where} \quad \delta^* = \delta/R_1
\]

(21)
and

$$\delta^* = \frac{A}{\sqrt{(K - (\omega^*)^2 + (C \omega^*)}^2} \quad \phi = \tan^{-1}\left[\frac{C \omega^*}{K - (\omega^*)^2}\right]$$

$$C = \frac{-M^* \omega^* \text{Im}(H)}{1 + M^* \text{Re}(H)} \quad K = \frac{1}{1 + M^* \text{Re}(H)} \quad A = \frac{e^*(\omega^*)^2}{1 + M^* \text{Re}(H)}$$

$\delta^*$ is the maximum dimensionless amplitude.

**NUMERICAL ANALYSIS, DISCUSSIONS AND EXPERIMENTAL VERIFICATION**

In the ongoing investigation, the specimen, being used, has got the following specifications.

1. Mild Steel cantilever type shaft rotating in viscous fluid (finite region) having a disk at the end span.
2. Length of the cantilever shaft, '$L$' = varies (1.0, 1.2, 1.5 m).
3. Radius of the cantilever shaft, '$R_i$' = 0.01 m.
4. Radius of the disk, '$R_{\text{Disk}}$' = 0.04 m.
5. Length of the disk '$L_{\text{Disk}}$' = 0.04 m.
6. Mass of the disk '$M_{\text{Disk}}$' = 1.57 kg.
7. Damping Co-efficient, '$\nu$' = varies (2.3, 0.427 & 0.0633 Stokes)
8. Equivalent mass of fluid displaced/corresponding mass of the shaft, '$m$' = varies (0.158, 0.1534 & 0.144)
9. Gap ratio, '$q_i$' = (4.0, 8.0, 12.0 & 20.0)

Fig.1a. shows the rotor with disk rotating in finite fluid region. Fig. 1b. shows the whirling motion of the rotor during rotating condition.

Numerical analysis for the system is carried out and the results obtained from numerical analysis for various aspects (e.g. frequency ratio vs. dimension amplitude ratio, virtual mass effect, gap ratio) are plotted in Figs 2 to 6.

The effect of rotating speed on amplitude of vibration are presented in Figs. 2 and 3 in dimensionless form. It is observed that as the viscosity of the external fluid increases, there is a shift in critical speed and decrease in amplitude of vibration.

From Fig.4 of the ongoing investigation, the container effect is highlighted. With the increase in container (fluid filled) radius, the corresponding dimensionless amplitude increases.

The effect of virtual mass damping can be observed from Figs.5 and 6. From Fig.5, it is noticed that with the increase in virtual mass effect (Im Re($H$)) at a constant damping effect, $(1 - m\omega \text{Im}(H))$, the critical speed of the rotor shifts towards left and the corresponding dimensionless amplitude of vibration decreases.

Due to damping effect (at constant virtual mass), the dimensionless amplitude gets affected in a more prominent way. With the increase in damping, the amplitude of vibration for the corresponding system decreases, which can be observed from Fig.6.
CONCLUSIONS

With respect to the ongoing analysis for the cantilever type rotor system in finite fluid region, the following conclusions are depicted:

1. Due to increase in co-efficient of viscosity, the amplitude of vibration decreases. Also critical speed shifts towards left.

2. As the gap ratio increases (container radius \( R_2 \) increases) the amplitude of vibration increases because of decrease in virtual mass effect \( |\text{Im}(H)| \) and the damping effect \( |\text{Re}(H)| \).

3. The virtual mass effect \( |\text{Im}(H)| \) shifts the critical speed towards left and also decreases the amplitude of vibration. But the damping effect \( |\text{Re}(H)| \), reduces the amplitude of vibration.

The above investigation can be used for vibration analysis of high speed shaft in viscous fluid, analysis of rotating shaft in high speed centrifuges and condition monitoring of dynamic rotor systems. The study can be extended for analysis of vibration characteristics of cracked rotor in viscous fluid.

NOTATIONS

\( D \) Diameter of the shaft \\
\( E \) Modulus of elasticity of shaft material 

\( F_x, F_y \) Fluid forces on rotor in x- and y-direction respectively \\
\( I \) Section moment of inertia of the shaft \\
\( L \) Total length of the shaft \\
\( M^* \) Equivalent mass of rotor \( (M_d + a_{eq} M_2) \) \\
\( M \) Dimensionless parameter \( (M / M_d) \) \\
\( M_r \) Equivalent mass of fluid displaced by a rotor \( (M_d + a_{eq} M_2) \) \\
\( R_1 \) Radius of the shaft \\
\( R_2 \) Radius of the cylinder \\
\( m \) Mass of the fluid displaced by the shaft per unit length \( (\pi R_2^2) \) \\
\( m_s \) Mass of the shaft per unit length \\
\( m^* \) Dimensionless parameter \( (m / m_s) \) \\
\( q_1 \) Gap ratio \( (R_2 - R_1) / R_1 \) \\
\( t \) Time \\
\( \alpha \) \( k R_1 \) \\
\( \beta \) \( k R \) \\
\( \delta \) Whirling radius of the shaft \\
\( \varepsilon \) Eccentricity \\
\( \mu \) Co-efficient of viscosity \\
\( \nu \) Co-efficient of kinematic viscosity \\
\( \rho \) Fluid density \\
\( \omega_{or} \) Natural angular frequency of the uncracked rotor in air \\
\( \omega_1 \) Fundamental natural frequency of the shaft (Without disk)

REFERENCES


Dynamic deflection of a cracked beam with moving mass

D R Parhi and A K Behera
Department of Mechanical Engineering, Regional Engineering College, Rourkela, Orissa, India

Abstract: An analytical method along with the experimental verification have been utilized to investigate the vibrational behaviour of a cracked beam with a moving mass. The local stiffness matrix is taken into account when analysing the cracked beam. The Runge-Kutta method has been used to solve the differential equations involved in analysing the dynamic deflection of a cantilever beam.

Keywords: crack, moving mass, Runge-Kutta method

NOTATION

\( a \)   crack depth
\( A \)   beam cross-sectional area
\( B \)   width of the beam
\( E \)   modulus of elasticity of beam material
\( F(t) \)   load due to moving mass \( M \)
\( g \)   acceleration due to gravity
\( I \)   section moment of inertia of the beam
\( L \)   total length of the beam
\( L_i \)   crack position
\( m \)   mass per unit length of the beam
\( M \)   lumped mass of the moving body
\( n \)   integer variable (varies from 1 to 8)
\( P(x, t) \)   external load on the beam due to moving mass \( M \)
\( q \)   integer variable (varies from 1 to \( \infty \))
\( Q(t) \)   general loading
\( t \)   time
\( T_{n\theta}(t) \)   \( d^2T_{n\theta}(t)/dt^2 = d/dx \)
\( v \)   velocity of the moving mass
\( V_n \)   \( \int_0^L Y_n(x)Y_n(x) \, dx \)
\( W \)   height of the beam
\( x \)   distance of the point from the fixed end to the point of interest where deflection is desired
\( y \)   transverse dynamic deflection of the beam
\( Y_n \)   eigenfunctions of the beam
\( \alpha_a \)   \( a \sqrt{W} \)
\( \beta \)   \( vt \)
\( \beta \beta \)   \( L_i/L \)
\( \gamma_n \)   eigenvalues
\( \delta \)   Dirac delta function
\( \rho \)   mass density of the beam

\( \omega_n \)   natural circular frequency of the beam for the \( n \)th mode

1 INTRODUCTION

For several decades, engineers have been investigating the potential hazard produced by moving masses on structures. The dynamic response of structures carrying moving masses is a problem of widespread practical significance. At the beginning of the twentieth century, engineers such as Jeffcott (1) managed to calculate the dynamic response of simple structures with a moving mass.

The investigations of Florence (2), Steele (3), Kenney (4) and Smith (5) were focused on determining response characteristics of a beam subjected to a moving force. Saigal (6) developed expressions for beam structures with the help of Stanisic \( et al. \) (7), which has a higher degree of practical significance. Later Akin and Mofid (8) analysed such problems for finite beams with moving loads using differential equations.

Parhi \( et al. \) (9) have discussed the vibrational analysis of a cantilever beam with a transverse crack using influence coefficients and stiffness elements at the crack section, and have verified their findings experimentally. Similarly, Papadopoulos and Dimurogonas (10) analysed theoretically the behaviour of a cracked shaft.

Despite the ever increasing number of research publications on the dynamic response of structures with moving masses, no significant investigation has been published on the dynamic response of beams with inherent cracks due to a moving mass. The presence of a crack is, of course, a major catalyst for structural failure.

In this paper a systematic theoretical approach has been developed for the dynamic response of a cracked beam with a moving beam along with experimental
verifications. The local stiffness matrix (9) is taken into account at the crack section to develop the theory.

2 EQUATION OF MOTION

The equation of motion of a uniform beam of mass \( m \) (mass per unit length) subjected to a moving mass \( M \), as shown in Fig. 1, can be written, neglecting damping, as

\[
EI \frac{d^4y(x, t)}{dx^4} + m \frac{d^2y(x, t)}{dt^2} = P(x, t)
\]  

(1)

The External force \( P(x, t) \) can be taken as

\[
P(x, t) = F(t) \delta(x - vt)
\]  

(2)

where

\[
F(t) = Mg - M \frac{d^2y(ut, t)}{dt^2}
\]  

(3)

where \( F(t) \) is the force due to the moving mass at that instant and \( \delta \) is the Dirac delta function (11).

From equations (1), (2) and (3),

\[
EI \frac{d^4y(x, t)}{dx^4} + \rho A \frac{d^2y(x, t)}{dt^2} = \left[Mg - M \frac{d^2y(\beta, t)}{dt^2}\right] \delta(x - \beta)
\]  

(4)

where

\[
\beta = ut
\]

\[
\rho A = m = \text{mass per unit length of the beam}
\]

\[
v = \text{velocity of the moving mass}
\]

\[
t = \text{time taken by the moving mass to travel a distance} \beta \text{ from the fixed end of the cantilever beam}
\]

Fig. 1 Beam with moving mass: \( z \) is the point of interest where the displacement of the beam is to be considered

\( x \) = distance of the point from the fixed end to \( z \), the point of interest where the deflection of the beam is considered (\( z \) it may be anywhere on the beam)

It is assumed that the solution of equation (4) is in a series form, that is

\[
y(x, t) = \sum_{n=1}^{\infty} Y_n(x) T_n(t)
\]  

(5)

Here \( Y_n(x) \) is the eigenfunction of the beam (without \( M \)) with the same boundary condition for the \( n \)th mode. To find out \( Y_n(x) \) the equation will be

\[
Y_n^{\text{ini}}(x) - \gamma_n^2 Y_n(x) = 0
\]  

(6)

where

\[
\gamma_n^2 = \frac{\rho A}{EI} \alpha_n^2
\]

and \( \alpha_n, n = 1, 2, 3, \ldots \), are the natural frequencies of the beam.

The general solution of equation (6) can be written as

\[
Y_n(x) = a_1 \sin(\alpha_n x) + b_1 \cos(\alpha_n x) + c_1 \sinh(\alpha_n x) + d_1 \cosh(\alpha_n x)
\]

\[0 \leq x \leq L_1 \text{ before the crack region} \]  

(7a)

\[
Y_n(x) = a_2 \sin(\alpha_n x) + b_2 \cos(\alpha_n x) + c_2 \sinh(\alpha_n x) + d_2 \cosh(\alpha_n x)
\]

\[L_1 \leq x \leq L \text{ after the crack region} \]  

(7b)

where \( a_1, b_1, c_1, d_1, a_2, b_2, c_2 \) and \( d_2 \) are constant coefficients, depending upon the boundary conditions, and are evaluated as shown in Appendix 1.

\( T_n(t) \) in equation (5) is the function of time and can be calculated by rewriting the righthand side of equation (4) as

\[
\left[Mg - M \frac{d^2y(\beta, t)}{dt^2}\right] \delta(x - \beta) = \sum_{n=1}^{\infty} Y_n(x) S_n(t)
\]  

(8)

Now substituting equation (5) into equation (8), equation (8) transforms to

\[
\left\{Mg - M \frac{d^2[\sum_{n=1}^{\infty} Y_n(\beta) T_n(t)]}{dt^2}\right\} \delta(x - \beta)
\]  

\[= \sum_{n=1}^{\infty} Y_n(x) S_n(t)
\]

\[= \left\{Mg - M \left[\sum_{n=1}^{\infty} Y_n(\beta) T_n(t)\right]\right\} \delta(x - \beta)
\]

\[= \sum_{n=1}^{\infty} Y_n(x) S_n(t)
\]  

(9)
Multiplying both sides of equation (9) by \( Y_p(x) \) and integrating over the beam length gives

\[
\int_0^L MgY_p(x)\delta(x - \beta) \, dx = -M \sum_{n=1}^\infty \int_0^L Y_p(x)Y_n(\beta)T_{n,n}(t)\delta(x - \beta) \, dx
\]

\[
= \sum_{n=1}^\infty \int_0^L Y_p(x)Y_n(\beta)S_n(t) \, dx
\]

From the Dirac delta property (Appendix 2),

\[
Mg \int_0^L Y_p(x)\delta(x - \beta) \, dx = MgY_p(\beta)
\]

\[
M \sum_{n=1}^\infty \int_0^L Y_p(x)Y_n(\beta)T_{n,n}(t)\delta(x - \beta) \, dx
\]

\[
= M \sum_{n=1}^\infty Y_n(\beta)T_{n,n}(t) \int_0^L Y_p(x)\delta(x - \beta) \, dx
\]

\[
= M \sum_{n=1}^\infty Y_n(\beta)T_{n,n}(t)Y_p(\beta)
\]

From the orthogonality principle and from the orthogonal properties of the function \( Y_n(x) \), the right-hand side of equation (10) becomes

\[
\sum_{n=1}^\infty \int_0^L Y_p(x)Y_n(x)S_n(t) \, dx = \sum_{n=1}^\infty S_n(t) \int_0^L Y_p(x)Y_n(x) \, dx
\]

\[
= S_p(t) \int_0^L Y_p(x)Y_1(x) \, dx
\]

\[
+ S_2(t) \int_0^L Y_p(x)Y_2(x) \, dx + \ldots
\]

\[
+ S_p(t) \int_0^L Y_p(x)Y_p(x) \, dx + \ldots
\]

\[
= S_p(t) \int_0^L Y_p(x)Y_p(x) \, dx
\]

\[
= S_p(t)Y_p
\]

\[
= V_pS_p(t)
\]

Using expressions (11), (12) and (13), equation (10) becomes

\[
MgY_p(\beta) - M \sum_{n=1}^\infty Y_n(\beta)T_{n,n}(t)Y_p(\beta) = S_p(t)V_p
\]

Equation (14) can be rewritten as

\[
S_p(t) = \frac{M}{V_p} \left[ g - \sum_{n=1}^\infty Y_n(\beta)T_{n,n}(t) \right] Y_p(\beta)
\]

From equations (4) and (8),

\[
EI \frac{d^4y(x, t)}{dx^4} + \rho A \frac{d^2y(x, t)}{dt^2} = \sum_{n=1}^\infty Y_n(\beta)S_n(t)
\]

Combining equations (15) and (16) gives

\[
EI \frac{d^4y(x, t)}{dx^4} + \rho A \frac{d^2y(x, t)}{dt^2} = \sum_{n=1}^\infty Y_n(x)\frac{M}{V_n} \left[ g - \sum_{q=1}^\infty Y_q(\beta)T_{q,n}(t) \right] Y_n(\beta)
\]

From equations (5) and (17),

\[
EI \frac{d^4\left( \sum_{n=1}^\infty Y_n(x)T_{n,n}(t) \right)}{dx^4} + \rho A \frac{d^2\left( \sum_{n=1}^\infty Y_n(x)T_{n,n}(t) \right)}{dt^2} = \sum_{n=1}^\infty Y_n(x)\frac{M}{V_n} \left[ g - \sum_{q=1}^\infty Y_q(\beta)T_{q,n}(t) \right] Y_n(\beta)
\]

Equation (18) can be rewritten as

\[
EI \frac{d^4\left( \sum_{n=1}^\infty Y_n(x)T_{n,n}(t) \right)}{dx^4} + \rho A \frac{d^2\left( \sum_{n=1}^\infty Y_n(x)T_{n,n}(t) \right)}{dt^2} = \sum_{n=1}^\infty Y_n(x) \frac{M}{V_n} \left[ g - \sum_{q=1}^\infty Y_q(\beta)T_{q,n}(t) \right] Y_n(\beta)
\]

Equation (19) gives

\[
EI \sum_{n=1}^\infty Y_n(x)T_{n,n}(t) + \rho AT_{n,n}(t)
\]

Substituting the value of \( Y_n^*(x) \) from equation (6) into equation (19) gives

\[
EI \sum_{n=1}^\infty Y_n^*(x)T_{n,n}(t) + \rho AT_{n,n}(t)
\]

Equation (20) may be rearranged as

\[
\sum_{n=1}^\infty Y_n(x) \left\{ EI \gamma_n^2T_{n,n}(t) + \rho AT_{n,n}(t) \right\}
\]

\[
= \sum_{n=1}^\infty Y_n(x) \frac{M}{V_n} \left[ g - \sum_{q=1}^\infty Y_q(\beta)T_{q,n}(t) \right] Y_n(\beta) = 0
\]

As equation (21) must be satisfied, for any arbitrary value of \( x \), then

\[
\rho A\omega_n^2T_{n,n}(t) + \rho AT_{n,n}(t)
\]

Equation (22) is a differential equation of the second order, which can be solved by the Runge-Kutta method.
3 NUMERICAL ANALYSIS

Equation (4) is solved using equation (5). The values of $T_n(t)$ are obtained at the required position from equation (22) using the Runge-Kutta method. The first six modes of vibration are taken into account for convergence of the results to an order of $10^{-4}$ units.

A mild steel beam ($L = 1.5 \text{ m}$, $B = 0.05 \text{ m}$, $W = 0.006 \text{ m}$) of Young’s modulus $210 \text{ GN/m}^2$, with a transverse crack $0.003 \text{ m}$ deep ($a_1$), having crack width ($= B$) at a distance $0.1 \text{ m}$ from the fixed end of the beam, along with moving masses $M$ ($1.2$ and $1.848 \text{ kg}$), is taken for the numerical analysis.

Figures 2 to 9 show the variation of displacement at the free end of cracked as well as uncracked beams with variations of time for different velocities of the moving masses ($1.2$ and $1.848 \text{ kg}$).
4 EXPERIMENTAL SET-UP

The experimental set-up used for performing the experiments is shown in Fig. 10. A number of tests were conducted on a mild steel beam specimen (1.5 × 0.05 × 0.006 m) with a transverse crack of 0.003 m depth at a distance (L_i = 0.1 m) from the fixed end (and without crack) with moving masses M (1.2 and 1.848 kg). Experimental results of free end deflection for various positions of the moving mass were recorded by positioning the LVDT (linearly variable displacement transducer) at the free end. The LVDT was arranged in the measuring system so that it only recorded the instantaneous displacement of the beam in the digital displacement indicator as the moving mass reached a predetermined location on the cantilever beam specimen. The results for different velocities of the moving mass are plotted in Figs 11 to 14. Corresponding numerical results are also presented in the same graphs for comparison.

While conducting the experiments precautions were taken in positioning the LVDT and adjusting the variac accurately in order to achieve the desired constant velocities of moving masses and the best possible accuracy in measurements.
From the numerical analysis and experimental results, the following points are drawn to the reader's attention:

1. It is evident from Figs 2 to 9 that, as the velocity of the moving mass increases, the deflection at the end-point of the beam decreases. This is because, at higher speeds, lower modes are not in resonance, which mainly contributes to a larger transverse amplitude of the beam.

2. From Figs 2 to 9 it is also found that as the mass of the moving body increases, the end deflection of the cracked beam increases relative to that of the corresponding uncracked beam. However, in both cases, the deflection increases as the distance of the moving mass from the fixed end increases.

The points discussed above can also be visualized from the experimental results, that is from Figs 11 to 14.
Fig. 8  Displacement at the end-point (m) versus time (s). Mild steel beam (1.5 x 0.05 x 0.006 m), \( M = 1.848 \, \text{kg}, \, v = 60 \, \text{km/h} \)

Fig. 9  Displacement at the end-point (m) versus time (s). Mild steel beam (1.5 x 0.05 x 0.006 m), \( M = 1.848 \, \text{kg}, \, v = 180 \, \text{km/h} \)

6 CONCLUSIONS

From the theoretical and experimental analysis as well as the discussion points enumerated above, the following conclusions may be drawn from the present investigation.

1. The deflection of the beam structure mainly depends upon the velocity of the moving mass. As the moving mass velocity increases, the dynamic deflection of the beam decreases.
2. At higher moving mass velocities, the lower modes of vibration are not in resonance (which contributes to a larger amplitude of vibration of the beam), so that the deflection of the beam decreases. However, as the magnitude of the moving mass increases, the deflection increases.

3. It is also evident from the theoretical and experimental analysis that the presence of cracks makes a significant difference in dynamic deflection of the beam compared to that of an uncracked one.

4. The present study may be extended to provide information about the life-span of complicated structures with cracks used in practice due to a moving mass system.
Fig. 13 Displacement at the end-point (m) versus time (s). Mild steel beam (1.5 x 0.05 x 0.006 m), $M = 1.848$ kg, $v = 40$ km/h

Fig. 14 Displacement at the end-point (m) versus time (s). Mild steel beam (1.5 x 0.05 x 0.006 m), $M = 1.848$ kg, $v = 60$ km/h

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APPENDIX 1

Analysis of the vibration characteristics of a cracked beam

A cantilever beam of length $L$ width $B$ and height $W$, with a crack of depth $a_i$, at a distance $L_1$ from the fixed end, is considered (as shown in Fig. 1). $U_1(x, t)$ and $U_2(x, t)$ are taken as the displacement functions for longitudinal vibrations of the sections before and after the crack and $F_1(x, t)$ and $F_2(x, t)$ are the deflection functions for bending vibrations for the same sections.

The normalized functions for the system in dimensionless form may be defined as

\begin{align}
\bar{u}_1(\bar{x}) &= A_1 \cos(\bar{R}_1 \bar{x}) + A_2 \sin(\bar{R}_2 \bar{x}) \\
\bar{u}_2(\bar{x}) &= A_3 \cos(\bar{R}_1 \bar{x}) + A_4 \sin(\bar{R}_2 \bar{x}) \\
\bar{y}_1(\bar{x}) &= A_5 \cos(\bar{R}_1 \bar{x}) + A_6 \sin(\bar{R}_2 \bar{x}) + A_7 \cos(\bar{R}_2 \bar{x}) + A_8 \sin(\bar{R}_2 \bar{x}) \\
\bar{y}_2(\bar{x}) &= A_9 \cos(\bar{R}_1 \bar{x}) + A_{10} \sin(\bar{R}_2 \bar{x}) + A_{11} \cos(\bar{R}_2 \bar{x}) + A_{12} \sin(\bar{R}_2 \bar{x})
\end{align}

where

\begin{align}
\bar{x} &= \frac{x}{L}, & \bar{u} &= \frac{u}{L}, & \bar{y} &= \frac{y}{L} \\
\bar{t} &= \frac{t}{L}, & \beta \beta &= \frac{L_1}{L} \\
\bar{R} &= \frac{\omega L}{C_s}, & C_s &= \left(\frac{E}{\rho}\right)^{1/2}, & \bar{R}_2 &= \left(\frac{\omega L}{C_s}\right)^{1/2} \\
C_s &= \left(\frac{E}{\mu}\right)^{1/2}, & \mu &= A \rho
\end{align}

$A =$ shaft cross-section
$\rho =$ mass density of the material
$E =$ Young's modulus of elasticity
$A_i (i = 1, 12) =$ constants are to be determined; these constants will be determined from the boundary conditions

The boundary conditions of the cantilever beam in consideration are:

\begin{align}
\bar{u}_1(0) &= 0 \quad (24a) \\
\bar{y}_1(0) &= 0 \quad (24b) \\
\bar{y}_1(0) &= 0 \quad (24c) \\
\bar{\alpha}_1'(1) &= 0 \quad (24d) \\
\bar{y}_2'(1) &= 0 \quad (24e) \\
\bar{\rho}_2'(1) &= 0 \quad (24f)
\end{align}

At the cracked section:

\begin{align}
\bar{u}_1(\beta \beta) &= \bar{u}_2(\beta \beta) \quad (25a) \\
\bar{y}_1(\beta \beta) &= \bar{y}_2(\beta \beta) \quad (25b) \\
\bar{y}_1'(\beta \beta) &= \bar{y}_2'(\beta \beta) \quad (25c) \\
\bar{y}_2''(\beta \beta) &= \bar{y}_2''(\beta \beta) \quad (25d)
\end{align}

Also, at the cracked section,

\begin{align}
AE \frac{d^2 u_1(L_1)}{dx^2} &= K_{11} [u_2(L_1) - u_1(L_1)] \nonumber \\
&+ K_{12} \left[ \frac{dy_2(L_1)}{dx} - \frac{dy_1(L_1)}{dx} \right] \\
&\text{Multiplying both sides by } AE(LK_{11}K_{12}) \text{ and simplifying,} \\
M_1 M_2 \bar{u}_1(\beta \beta) &= M_2 [u_2(\beta \beta) - u_1(\beta \beta)] \\
&+ M_1 \bar{y}_2'(\beta \beta) - \bar{y}_1'(\beta \beta) \quad (26)
\end{align}

(The above equation is for discontinuity due to axial deformation, before and after the crack.) Similarly,

\begin{align}
EI \frac{d^2 y_1(L_1)}{dx^2} &= K_{21} [u_2(L_1) - u_1(L_1)] \nonumber \\
&+ K_{22} \left[ \frac{dy_2(L_1)}{dx} - \frac{dy_1(L_1)}{dx} \right] \\
&\text{Multiplying both sides by } EI(L^3 K_{21} K_{22}) \text{ and simplifying,} \\
M_3 M_4 \bar{y}_1(\beta \beta) &= M_3 [u_2(\beta \beta) - u_1(\beta \beta)] \\
&+ M_4 \bar{y}_2'(\beta \beta) - \bar{y}_1'(\beta \beta) \quad (27)
\end{align}

where

\begin{align}
M_1 &= \frac{AE}{LK_{11}}, & M_2 &= \frac{AE}{K_{12}} \\
M_3 &= \frac{EI}{LK_{21}}, & M_4 &= \frac{EI}{L^3 K_{22}}
\end{align}
The local flexibility matrix elements $K_{ij}$ are obtained by inverting the compliance matrix. (The above equation is for discontinuity in slope, before and after the crack of the beam.)

Compliance matrix elements $C_{ij}$ (influence coefficients) can be obtained by considering the strain energy (12) at the crack location.

$$C_{ij} = \frac{\delta^2}{\partial P_i \partial P_j} \int_0^{\pi} J(\alpha) \, d\alpha$$

$J(\alpha) = \text{strain energy density function}$

$P_i (i = 1, 2) = \text{axial force } (i = 1), \text{bending moment } (i = 2)$

APPENDIX 2

The Dirac delta function $\delta(x - \beta)$ has the following properties:

$$\delta(x - \beta) = \begin{cases} 0, & \text{for all } x \neq \beta \\ \text{greater than any assumed value,} & \text{for } x = \beta \end{cases}$$

Then for a beam of length $L$,

$$\int_0^{\pi} f(x) \delta(x - \beta) \, dx = \begin{cases} f(\beta), & 0 < \beta < L \\ 0, & \beta < 0, \beta > L \end{cases}$$
Technical Note

Vibrational analysis of a cracked shaft

Dayal R. Parhi* and A.K. Behera*

Evaluation of the characteristics of transverse vibration of elastic structures with crack is required for its continued use in many applications. Natural frequencies and mode shapes vary with location, size and boundary conditions. Adeli1 et al. used the compliance co-efficients to formulate the boundary conditions at the cracked portion. Adams2 et al. modelled the localized damage by a linear spring of very small length separating two sections of the bar. Akgun3 et al. developed the concept of 'fracture hinge' analytically to represent a cracked section to diagnose fracture damage in a simple structure. Ju and Mivorich4, verified the postulates of fracture hinge which were applied to a simple structure with small number of cracks.

Gudmundson5 considered a bar with free ends and a transverse crack at the center. He developed an equation by the help of perturbation method for the changes in natural frequencies, which compared well with the values obtained by the finite element analysis. Nian, Lin, Shing and An6 studied the variation of the vibration parameters after crack occurrence and used to identify structural fault.

Mayes and Davies7 studied this effect of a transverse crack on the vibrational behavior of a turbine rotor in order to develop the criteria for the detection of cracks in shafts of turbogenerators from the measurement of their dynamic response.

A systematic approach is presented in this note taking coupling of transverse and torsional vibrations into account, for evaluation of normal modeshapes and natural frequencies of elastic, simply supported shaft with transverse crack. It is observed that the natural frequencies of the cracked specimen decreases with the increase in relative crack depth and there is an abrupt change in mode shape, which can be useful for detection of crack.

LOCAL FLEXIBILITY OF A CRACKED SHAFT UNDER BENDING AND TORSIONAL LOADING

The presence of a transverse surface crack of depth 'a' on a shaft of radius 'R' and crack width '2b' introduces a local flexibility which can be derived from strain energy release rate. The strain energy release rate8 at the fractured section can be written as:

$$J = \frac{1}{E} \left[ (\sum_{i=1}^{6} K_{ij})^2 + (\sum_{i=1}^{6} K_{ij})^2 + m (\sum_{i=1}^{6} K_{ij})^2 \right]$$

(1)

where

$$\frac{1}{E} = \frac{1+\nu}{E}$$

$$\nu = \text{Poisson's Ratio}$$

$$E = \text{Young's Modulus of elasticity}$$

$$m = 1 + \nu$$

$$K_{ij} = \text{Stress intensity factors}$$

$$i = \text{Mode I, II, III opening of the crack and for load}$$

$$P_j, j = 1, 6.$$  

Let $U_i$ be the strain energy due to the crack. Then from Castigliano's theorem9, the additional displacement along the force $P_i$ is:

$$u_i = \frac{\partial U_i}{\partial P_i}$$

(2)

The strain energy will have the form

$$U_i = \int_0^a \frac{\partial U_i}{\partial a} da = \int_0^a J da$$

(3)

Where $J = \frac{\partial U_i}{\partial a}$ the strain energy density function.

From (1) and (2) we have

$$u_i = \frac{\partial}{\partial P_i} \left[ \int_0^a J (a) da \right]$$

(4)

* Professor. Mechanical Engineering Department, Regional Engineering College, Rourkela - 769 008, India
The local flexibility \((C_y)\) due to crack per unit of width can be written as:
\[
C_y = \frac{\partial u_j}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j} \int_0^a J(a) \, da \tag{5}
\]
To find the final flexibility matrix, integration is done over the crack width \(2b\).
\[
C_y = \frac{\partial u_j}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j} \int_{-b}^b \int_0^a J(a) \, da \, dz \tag{6}
\]
From Tada\(^8\) it can be written
\[
K_{II} = \left[4P_4 \frac{\gamma}{(\pi R^3)} \right] (\pi \alpha)^{0.5} F_I (\alpha/h) \tag{7}
\]
\[
K_{III} = \left[2P_6 \frac{\gamma}{(\pi R^3)} \right] (\pi \alpha)^{0.5} F_{II} (\alpha/h) \tag{8}
\]
\[
K_{IIV} = \left[\int_0^a \left(\gamma R^2 - \zeta^2\right)^{0.5}/(\pi R^3) \right] (\pi \alpha)^{0.5} F_{III} (\alpha/h) \tag{9}
\]
\[
K_{IV} = \left[2P_6 (R^2 - \zeta^2)^{0.5}/(\pi R^3) \right] (\pi \alpha)^{0.5} F_{IV} (\alpha/h) \tag{10}
\]
where
\[
F_I (\alpha/h) = \left(\tan \lambda/\lambda\right)^{0.5} \left(0.752 + 2.02 \alpha/h + 0.37(1 - \sin \lambda)^3 \right) / (1 - \alpha/h)^{0.5} \tag{11}
\]
\[
F_{II} (\alpha/h) = [1.122 - 0.561 (\alpha/h) + 0.085 (\alpha/h)^3] + 0.18(\alpha/h)^3/(1 - \alpha/h)^{0.5} \tag{12}
\]
\[
F_{III} (\alpha/h) = (\tan \lambda/\lambda)^{0.5} \tag{13}
\]
From the above equations, the dimensionless compliance can be written as:
\[
\overline{C}_{22} = \frac{\pi ERC_{22}}{1 - \nu^2} = 4 \int_0^{\frac{h}{2}} \int_0^{\frac{b}{2}} \gamma F_{II} (\overline{h}) \, d\overline{z} \, d\overline{y} \tag{14}
\]
\[
\overline{C}_{44} = \frac{\pi ERC_{44}}{1 - \nu^2} = 64 \int_0^{\frac{h}{2}} \int_0^{\frac{b}{2}} \gamma F_{IV} (\overline{h}) \, d\overline{z} \, d\overline{y} \tag{15}
\]
\[
\overline{C}_{66} = \frac{\pi ERC_{66}}{1 - \nu^2} = 8 \int_0^{\frac{h}{2}} \int_0^{\frac{b}{2}} (1 - \zeta^2)^{0.5} \gamma F_{IV} (\overline{h}) \, d\overline{z} \, d\overline{y} \tag{16}
\]
\[
\overline{C}_{66} = \frac{\pi ERC_{66}}{1 - \nu^2} = 16 \int_0^{\frac{h}{2}} \int_0^{\frac{b}{2}} (a_1 + m a_2) \, d\overline{z} \, d\overline{y} \tag{17}
\]
where \(a_1 = \zeta^2 \gamma F_{IV} (\overline{h})\), \(a_2 = (1 - \zeta^2) \gamma F_{IV} (\overline{h})\) and \(\zeta = \gamma/R, \gamma = y/R, \lambda = \pi \alpha/2h, \overline{h} = h/R, \overline{b} = b/R\).
\[\overline{a} = \alpha/R, \alpha = h/2 - \gamma, h = 2(R^2 - \gamma^2)^{0.5}\]

The dimensionless compliance matrix can be written as:
\[
\overline{C} = \begin{bmatrix}
\overline{C}_{22} & \overline{C}_{26} & 0 \\
\overline{C}_{62} & \overline{C}_{66} & 0 \\
0 & 0 & \overline{C}_{44}
\end{bmatrix}
\tag{18}
\]

The local stiffness matrix can be obtained by taking the inversion of compliance matrix.
\[
K = \begin{bmatrix}
k_{22} & k_{26} & 0 \\
k_{62} & k_{66} & 0 \\
0 & 0 & k_{44}
\end{bmatrix} = \begin{bmatrix}
\overline{C}_{22} & \overline{C}_{26} & 0 \\
\overline{C}_{62} & \overline{C}_{66} & 0 \\
0 & 0 & \overline{C}_{44}
\end{bmatrix}^{-1}
\tag{19}
\]

Fig. 2 shows the variation of dimensionless compliance with the crack depth.

**ANALYSIS OF VIBRATION CHARACTERISTIC OF THE CRACKED SHAFT**

A simply supported shaft of length \(L\) radius \(R\), with a crack of depth \(V\) at a distance \(L_j\) from left end and crack width \(2b\) is considered (Fig.1). Taking \(\theta_1 (x, t)\) and \(\theta_2 (x, t)\) as deflection of torsional vibrations for the sections before and after the crack \(Y_1 (x), Y_2 (x)\), are the deflection for bending vibrations and \(\xi_1 (x, t), \xi_2 (x, t)\) are the slope of the deflection curve for the same sections, the normal functions in dimensionless form can be written as:

\[
\overline{\theta}_1 (x) = A_1 \, \text{CSK} + A_2 \, \text{SSK}
\]
\[
\overline{\theta}_2 (x) = A_3 \, \text{CSK} + A_4 \, \text{SSK}
\]
\[
\overline{Y}_1 (x) = a_1 \, \text{SSL} + b_1 \, \text{CSL} + c_1 \, \text{SSH} + d_1 \, \text{CSH}
\]
\[
\overline{Y}_2 (x) = a_2 \, \text{SSL} + b_2 \, \text{CSL} + c_2 \, \text{SSH} + d_2 \, \text{CSH}
\]
\[
\overline{\xi}_1 (x) = a_1 \, \text{CSL} - b_1 \, \text{SSL} + c_1 \, \text{CSH} + d_1 \, \text{SSH}
\]
\[
\overline{\xi}_2 (x) = a_2 \, \text{CSL} - b_2 \, \text{SSL} + c_2 \, \text{CSH} + d_2 \, \text{SSH}
\]

\[\overline{a} = \alpha/R, \alpha = h/2 - \gamma, h = 2(R^2 - \gamma^2)^{0.5}\]
where
\[
\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{\mu} = A \rho, \quad \text{CSK} = \cos (\bar{K}_u \bar{x})
\]

SSK = sin (\bar{K}_u \bar{x}), SSL = sin (\bar{\lambda}_1 \bar{x}), CSL = \cos (\bar{\lambda}_1 \bar{x})

COSH = \cosh (\bar{\lambda}_2 \bar{x}), \quad \text{SSH} = \sinh (\bar{\lambda}_2 \bar{x})

\[
\bar{K}_u = \frac{\omega L}{C_u}, \quad C_u = \left( \frac{G}{\rho} \right)^{\frac{1}{2}}
\]

\[
\lambda_0^2 = \frac{\mu L^2 \omega^2}{A E} (1 + \frac{E}{k G^2}), \quad \psi^2 = \frac{\mu L^4 \omega^2}{E I} (1 - \frac{E}{k G^2})
\]

\[
\lambda_{1,2} = (\psi^2 \pm \frac{\lambda_{0}^2}{4} + \frac{\lambda_{1,2}^2}{2})^{0.5}
\]

\[
\lambda_1 = (\lambda_1 - \lambda_2)/\lambda_1, \quad \lambda_2 = (\lambda_2 + \lambda_3)/\lambda_2
\]

\[
\lambda_3 = \omega/[(kGA/\mu L^2)]^{0.5}
\]

\[ A = \text{Shaft Cross-Section} \]

\[ \rho = \text{Mass density of the material} \]

\[ G = \text{Shear Modulus} \]

\[ k = 6(1 + v)/(7 + 6v) \]

\[ A_1, A_2, A_3, A_4, a_1, b_1, c_1, d_1, a_2, b_2, c_2 \text{ and } d_2 \text{ are the constants to be determined from the boundary conditions.} \]

The boundary conditions of the simply supported shaft in consideration are:

\[
Y_1'(0) = 0, \quad EI \xi'_1(0) = 0
\]

\[
Y_2(L) = 0, \quad EI \xi'_2(L) = 0
\]

\[
EI \xi'_1(L_1) = EI \xi'_2(L_1) = k_{44} (\xi'_2(L_1) - \xi'_1(L_1))
\]

\[
kGA (Y'_1 (L_1) - \xi_1 (L_1)) = kGA (Y'_2 (L_1) - \xi_2 (L_1))
\]

\[
= k_{22} (Y_2 (L_1) - Y_1 (L_1)) + k_{26} (\theta_2 (L_1) - \theta_1 (L_1)) \quad (21)
\]

\[
G I_p \theta'_1(0) = G I_p \theta'_2(L) = 0
\]

\[
G I_p \theta'_1 (L_1) = G I_p \theta'_2 (L_1)
\]

\[
= k_{62} (Y_2 (L_1) - Y_1 (L_1)) + k_{66} (\theta_2 (L_1) - \theta_1 (L_1))
\]

The normal functions (Eq.20) along with the above boundary conditions (Eq.21), yield the characteristic equation of the system.

\[
|Q| = 0 \quad (22)
\]

This determinant is a function of natural circular frequency (\(\omega\)), the location of the crack (\(L_1\)) and the local stiffness matrix (\(K\)) which is a function of the crack depth (\(a\)).

**NUMERICAL ANALYSIS AND DISCUSSION**

From equation (21) the natural frequencies for the first three modes are found out and subsequently the numerical analysis is carried out for simply supported Mild Steel shaft having length (\(L = 1.5m\)) and radius (\(R = 0.1m\)) for various relative crack depth (\(a/D\)) and relative crack position (\(L_1/L\)).

In Fig. 2, comparison is made among dimensionless compliance and relative crack depth (\(a/D\)) Variation of dimensionless frequency (\(\omega_{crack}/\omega_{uncrack}\)) to that of relative crack depth (\(a/D\)), keeping the relative crack position (\(L_1/L = 0.01\)) constant (Fig.3).

In Figs. 4 to 12 comparisons are made between mode shapes of uncracked and cracked shaft. Significant variation is noticed between the mode shapes of the uncracked and cracked shaft, particularly at the crack positions. In Fig.8 there is no change in mode shape because the point of inflection and location of the crack occurs at the same point of the shaft (mid point). It is also observed that as the relative crack depth increases, the variation in mode shape increases.
Relative Amplitude vs. Relative Distance from Left End of the Simply Supported MS Shaft (1st. Mode of Vibration), \( a/D = 0.3, L_1 = 40 \text{ cm}, L = 150 \text{ cm} \), \( D = 2\text{ cm} \), without crack, with crack.

Relative Amplitude vs. Relative Distance from Left End of the Simply Supported MS Shaft (1st. Mode of Vibration), \( a/D = 0.5, L_1 = 40 \text{ cm}, L = 150 \text{ cm} \), \( D = 2\text{ cm} \), without crack, with crack.

Relative Amplitude vs. Relative Distance from Left End of the Simply Supported MS Shaft (1st. Mode of Vibration), \( a/D = 0.3, L_1 = 75 \text{ cm}, L = 150 \text{ cm} \), \( D = 2\text{ cm} \), without crack, with crack.

Relative Amplitude vs. Relative Distance from Left End of the Simply Supported MS Shaft (1st. Mode of Vibration), \( a/D = 0.3, L_1 = 75 \text{ cm}, L = 150 \text{ cm} \), \( D = 2\text{ cm} \), without crack, with crack.
CONCLUSIONS

The conclusions obtained from this study are:

From Fig.3 it is evident that due to the presence of crack the resonant frequency of the shaft is lower than that of uncracked one. It is also observed that as the crack depth increases the resonant frequency decreases sharply.

Due to the presence of crack there are remarkable changes in mode shapes of the shaft when compared with the uncracked one. It is also observed that, at the location of the crack the mode shapes show abrupt changes (Figs.4 to 12). It is also noticed that with increase in crack depth, variation in mode shapes are pronounced.

All these information regarding vibration can be used to locate and identify crack in different structures. Further study is on to improve and apply this method for various practical field and also for structures having multi-cracks.

REFERENCES


THE DYNAMICS OF A ROTOR SYSTEM WITH A CRACKED SHAFT IN VISCOUS MEDIUM

Dayal R. Parhi* and A.K. Behera**

Abstract

A theoretical analysis of the dynamics of a cracked rotor with a disc rotating in a viscous fluid is presented. Virtual mass effect and viscous damping effects are determined applying dynamic forces due to viscous fluid on the rotating cracked shaft with the help of Navier Stokes equations. Influence co-efficients at the cracked section are taken into account for finding the natural frequencies of vibration and stiffness of the cracked shaft. It is observed that due to presence of the crack and viscous fluid, there are changes in critical speeds and amplitude of vibrations of rotating shaft.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>Crack depth</td>
</tr>
<tr>
<td>(b)</td>
<td>Half the width of the crack</td>
</tr>
<tr>
<td>(D_1)</td>
<td>Diameter of the shaft</td>
</tr>
<tr>
<td>(\delta)</td>
<td>Whirling radius of the shaft</td>
</tr>
<tr>
<td>(E)</td>
<td>Modulus of elasticity of shaft material</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td>Eccentricity</td>
</tr>
<tr>
<td>(F_x, F_y)</td>
<td>Fluid forces on rotor in x-and y-direction respectively</td>
</tr>
<tr>
<td>(I)</td>
<td>Section moment of inertia of the shaft</td>
</tr>
<tr>
<td>(K_{55}, K_{44})</td>
<td>Stiffness of the cracked shaft in two directions (55- and 44-)</td>
</tr>
<tr>
<td>(L)</td>
<td>Total length of the shaft</td>
</tr>
<tr>
<td>(L_i)</td>
<td>Crack position from left side of the shaft</td>
</tr>
<tr>
<td>(m)</td>
<td>Mass of the fluid displaced by the shaft per unit length ((\rho \pi R_1^2))</td>
</tr>
<tr>
<td>(m_s)</td>
<td>Mass of the shaft per unit length</td>
</tr>
<tr>
<td>(m^*)</td>
<td>Dimensionless parameter (=(m/m_s))</td>
</tr>
<tr>
<td>(M_{1M})</td>
<td>Equivalent mass of the fluid displaced by rotor</td>
</tr>
<tr>
<td>(M_{2M})</td>
<td>Equivalent mass of the fluid displaced by rotor</td>
</tr>
<tr>
<td>(q_1)</td>
<td>Gap ratio ((R_2-R_1)/R_1)</td>
</tr>
<tr>
<td>(R_1)</td>
<td>Radius of the shaft</td>
</tr>
<tr>
<td>(R_2)</td>
<td>Radius of the cylinder</td>
</tr>
<tr>
<td>(\tau)</td>
<td>Time</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>= (k R_1)</td>
</tr>
<tr>
<td>(\alpha\alpha)</td>
<td>Relative crack position ((L_L/L))</td>
</tr>
<tr>
<td>(\beta)</td>
<td>= (k R_2)</td>
</tr>
<tr>
<td>(\beta\beta)</td>
<td>Relative crack depth ((a_1/D_1))</td>
</tr>
<tr>
<td>(\mu)</td>
<td>Co-efficient of viscosity</td>
</tr>
<tr>
<td>(\nu)</td>
<td>Co-efficient of kinematic viscosity</td>
</tr>
</tbody>
</table>

\(\rho\) = Fluid density

\(\omega_0\) = Natural angular frequency of the uncracked rotor in air

\(\omega_{\text{crack}}\) = Natural angular frequency of the cracked rotor

Introduction

Natural frequencies and mode shapes are important dynamic characteristics of a structure as they are required in the design of rotating shafts. Determination of critical speed of shafts is quite common using established standard procedures. However when a shaft rotates in a different media other than ambient air having a crack, the crack as well as the viscosity of media play important roles in the determination of critical speed.

In order to study the effect of fluid forces acting on a rotor in a liquid, Kito [1] analysed an eccentrically rotating circular rod in a circular cylinder with the assumption that the flow velocity distributes linearly over the gap between the rod and the cylinder. Iida [2] also treated a circular rod rotating in an infinitely extending water region, but the influence of liquid viscosity and the gap were not discussed in greater detail. Further, Fritz [3] analysed the similar problem taking account of the influence of turbulence and Taylor vortex for a very small rod-cylinder gap. Brenner [4] analysed theoretically the fluid forces acting on a cylinder for high and Reynold's number.

* Lecturer ** Professor

Department of Mechanical Engineering, Regional Engineering College, Rourkela, Orissa, India

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Parhi and Behera [5] in their paper discussed the vibrational behavior of a cracked beam, with the help of influence coefficients and stiffness of the cracked shaft. Papadopoulos et.al.[6] analysed the bending vibration of a rotating shaft with a crack, but the influence of liquid viscosity, virtual mass effect and gap ratio were not extensively discussed.

In this paper vibrational analysis of a cracked rotor in a viscous fluid is carried out. The effect of viscosity, virtual mass and gap ratio on the behavior of cracked rotor are taken into consideration. With the help of the Navier Stokes equation, the influence co-efficients and stiffness matrix at the crack section under the above effects are found out. Variations are observed in critical speeds and amplitudes of vibrations when comparisons are made for cracked and uncracked shafts.

Analysis of Fluid Motion

A cracked shaft of cross sectional radius \( R_1 \) is rotating with a speed \( \omega \) having whirling speed \( \Omega \) with \( \delta \) as the radius, is shown in Fig.1.

\[ \frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nabla \left( \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \frac{u}{r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r} \frac{\partial v}{\partial r} \right) + \frac{1}{\rho} \frac{\partial p}{\partial \theta} + \nabla \left( \frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} \frac{v}{r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} - \frac{2}{r} \frac{\partial u}{\partial r} \right) \]  

Equation (2) can be divided into two parts i.e.

\( \nabla^4 \psi - \frac{1}{v} \frac{\partial}{\partial t} \nabla^2 \psi = 0 \)  

where \( \nabla^2 \psi = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \)

Equation (2) can be divided into two parts i.e.

\[ \nabla^2 \psi = 0, \quad \nabla^2 \psi - \left( \frac{1}{v} \right) \left( \frac{\partial w}{\partial t} \right) = 0 \]  

In the above equation \( u \) and \( v \) denote the flow velocity in radial and tangential direction respectively and \( p \) is the fluid pressure. With the help of the stream function \( \psi(r, \theta, t) \) and eliminating the pressure terms \([4, 13]\), the above equations can be written as.

\[ \nabla^4 \psi - \frac{1}{v} \frac{\partial}{\partial t} \nabla^2 \psi = 0 \]  

\[ \nabla^2 \psi = 0, \quad \nabla^2 \psi - \left( \frac{1}{v} \right) \left( \frac{\partial w}{\partial t} \right) = 0 \]
The solution of equation (2) is expressed as:

\[ \Psi = \Psi_1 + \Psi_2 \]

where \( \Psi_1 \) and \( \Psi_2 \) are solutions of equation (3).

The radial and tangential components of flow velocity at point A in Fig. 1 are,

\[ u_0 = R_1 \omega \sin (\alpha) - \delta \Omega \sin (\Omega t - \theta) \]
\[ v_0 = R_1 \omega \cos (\alpha) + \delta \Omega \cos (\Omega t - \theta) \]

where \( \delta \) is the angle between \( O'A \) and \( OA \)

\[ \sin(\alpha) = \left( \frac{\delta}{R_1} \right) \sin (\Omega t - \theta) \text{ and } \cos(\alpha) = 1 \text{ for } \delta \ll R_1 \]

for \( r = R_1 \), the equation (4) can be rewritten as:

\[ u \mid _{r=R_1} \delta (\omega - \Omega) \sin (\Omega t - \theta) + \text{Re} \left[ \delta \omega e^{i(\omega t - \theta)} \right] + R_1 \omega \]

\[ v \mid _{r=R_1} \delta \Omega \cos (\Omega t - \theta) + R_1 \omega = \text{Re} \left[ (\delta \Omega e^{i(\Omega t - \theta)}) + R_1 \omega \right] (5) \]

where \( i = \sqrt{-1} \) and \( \text{Re} [.] \) denotes the real part of [.] .

Taking \( \omega = \Omega \) equations (5) reduces to ;

\[ u \mid _{r=R_1} = 0 \text{ , } v \mid _{r=R_1} = \text{Re} \left[ \delta \omega e^{i(\omega t - \theta)} \right] + R_1 \omega (6) \]

When the shaft is immersed in a finite extending fluid region the boundary conditions \( \{ r \to R_2 \} \) i.e. the container radius is taken as \( R_2 \) are taken as

\[ u \mid _{r=R_2} = v \mid _{r=R_2} = 0 \]

Under these conditions, non stationary components of the solutions \( \Psi_1 \) and \( \Psi_2 \) can be expressed as;

\[ \Psi_1 (r, \theta, t) = F_1 (r) e^{i(\omega t - \theta)} \]
\[ \Psi_2 (r, \theta, t) = F_2 (r) e^{i(\omega t - \theta)} \]

From equation (8) and equation (3) we obtain:

\[ \frac{d^2 F_1}{dr^2} + \frac{1}{r} \frac{d F_1}{dr} - \frac{1}{r^2} F_1 = 0 \] (9a)

\[ \frac{d^2 F_2}{dr^2} + \frac{1}{r} \frac{d F_2}{dr} - \left( \frac{1}{r^2} + k^2 \right) F_2 = 0 \] (9b)

where \( k = \sqrt{i\omega/\nu} \)

Equation (9a) is Euler’s equation and equation (9b) is Bessel’s equation. The solution to the above equations can be written as ;

\[ F_1 (r) = \delta \omega (AR_1^2/r + Br) \]
\[ F_2 (r) = \delta \omega R_1 (CI_1 (kr) + DK_1 (kr)) \] (10)

where \( A, B, C \) and \( D \) are arbitrary constants and \( I_1 (kr) \) and \( K_1 (kr) \) are modified Bessel’s functions of 1st and 2nd kind respectively.

The non-stationary components of flow velocities can be written as;

\[ u_1 = -\frac{1}{r} \frac{\partial v}{\partial \theta} = i \delta \omega \left[ A \left( \frac{R_1}{r} \right)^2 + B + C \frac{R_1}{r} \right] I_1 (kr) + D \frac{R_1}{r} K_1 (kr) \]
\[ v_1 = \frac{\partial v}{\partial r} = \delta \omega \left[ -A \left( \frac{R_1}{r} \right)^2 + B \right] + C \left[ -\frac{R_1}{r} I_1 (kr) - kR_1 I_0 (kr) \right] + D \left[ -\frac{R_1}{r} K_1 (kr) - kR_1 K_0 (kr) \right] (11) \]

Applying the boundary conditions (6) and (7) to Eq. 11, we obtain the equation for arbitrary constants, from where these can be found out, as follows,

\[ \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 & 1 & I(\alpha) \\ -1 & 1 & -I_1 (\alpha) + \alpha \gamma_0 (\alpha) \\ \gamma_1 \gamma_2 & 1 & \gamma_1 I_1 (\beta) \\ -\gamma_2^2 & 1 & -\gamma_1 I_1 (\beta) + \alpha \gamma_0 (\beta) \\ 1 & 0 & K_1 (\beta) - \alpha K_0 (\beta) \end{bmatrix} \] (11a)

where \( \alpha = kR_1, \beta = kR_2, \gamma \gamma R_1/R_2 \)

Analysis of Fluid Forces

Substituting the flow velocities given by equation (11) into equation (1), the non-stationary component of pressure \( p \) can be written as;

\[ p = \frac{1}{\gamma \gamma} \frac{\partial p}{\partial \theta} = \delta \rho \omega^2 \left[ A \left( \frac{R_1}{r} \right)^2 + B \right] (12) \]
Normal stress \( \tau_r \) and tangential stress \( \tau_\theta \) due to flow can be written as:

\[
\tau_r = -p + 2\mu \frac{\partial u}{\partial r} \quad \text{and} \quad \tau_\theta = \mu \left[ r \frac{\partial (\frac{\partial u}{\partial r})}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right]
\]  

Fluid forces acting on the surface (i.e., \( r=R_i \)) per unit length of the shaft in the \( x \) and \( y \) direction are written as:

\[
F_x = \int_0^{2\pi} (\tau_r \cos \theta - \tau_\theta \sin \theta) R_1 d\theta
= m \omega^2 [A - B - CI_1(\alpha) - DK_1(\alpha)] e^{i\omega t}
\]

\[
F_y = \int_0^{2\pi} (\tau_r \sin \theta + \tau_\theta \cos \theta) R_1 d\theta
= -i \pi \delta \omega^2 [A - B - CI_1(\alpha) - DK_1(\alpha)] e^{i\omega t}
\]

where \( \alpha = kR_i \), \( m = \pi R_i^2 \)  

Only the real parts of equation (14) has the meaning. So \( F_x \) and \( F_y \) after simplification can be expressed as

\[
F_x = m \delta \omega^2 \Re (H) \cos (\omega t) - \Im (H) \sin (\omega t)
\]

\[
F_y = m \delta \omega^2 \Re (H) \sin (\omega t) + \Im (H) \cos (\omega t)
\]

where \( H = A - B - CI_1(\alpha) - DK_1(\alpha) \) and \( \Re(H) \), \( \Im(H) \) denote the real and imaginary part of \( H \).

The coordinates of the center of the shaft (as shown in Fig.1) is \( x = \delta \cos \omega t \), \( y = \delta \sin \omega t \). Now

\[
F_x = -m \Re(H) \frac{d^2x}{dt^2} + m \omega \Im(H) \frac{dx}{dt}
\]

\[
F_y = -m \Re(H) \frac{d^2y}{dt^2} + m \omega \Im(H) \frac{dy}{dt}
\]

In equation (16), \( m \Re(H) \) denotes the virtual or added mass of the fluid relating to the inertia force of the shaft while \( -m \omega \Im(H) \) denotes the viscous damping coefficient.

Analysis of Rotor Behavior

Here a lumped mass at the mid span of the simply supported rotating cracked shaft immersed in the finite fluid region is considered.

Equivalent lumped masses of a rotating cracked shaft are given by \( K_{55}/\omega^2 + K_{44}/\omega^4 \) (in two directions as shown in Fig.2), where \( K_{55} \), \( K_{44} \) and \( \omega_{55} \), \( \omega_{44} \) are the stiffnesses and fundamental natural frequencies in those directions (see Appendix) respectively.

The ratio of the equivalent lumped mass to the total mass of the shaft in two main directions are given by the expressions;

\[
\alpha_{eq1} = \frac{K_{44}}{\omega_{44}^2 M_2} \quad \alpha_{eq2} = \frac{K_{55}}{\omega_{55}^2 M_2}
\]

where \( M_2 \) is the mass of the shaft.

If a disc with mass \( M_{1s} \) is attached at the mid span of the shaft, a total lumped mass of the rotor in 44- and 55-directions become

\[
M_1M_5 = M_{1s} + \alpha_{eq1} M_2 \quad M_2M_5 = M_{2s} + \alpha_{eq2} M_2
\]

The equations of motion of two equivalent single degree of freedom systems of the whirling cracked rotor in fluid are reduced to;

\[
M_1M_5 \frac{d^2x}{dt^2} (x + \varepsilon \cos \omega t) + K_{44} x = F_x
\]

\[
M_2M_5 \frac{d^2y}{dt^2} (y + \varepsilon \sin \omega t) + K_{55} y = F_y
\]

The fluid forces from equation (16) can be written as;

\[
F_x = -M_1M \Re(H) \frac{d^2x}{dt^2} + M_1M \omega \Im(H) \frac{dx}{dt}
\]

\[
F_y = -M_2M \Re(H) \frac{d^2y}{dt^2} + M_2M \omega \Im(H) \frac{dy}{dt}
\]

where

\[
M_1M \Re(H) = M_1 \Re(H_1) + \alpha_{eq1} M_2 \Re(H_2)
\]

\[
M_2M \Re(H) = M_1 \Re(H_1) + \alpha_{eq2} M_2 \Re(H_2)
\]

\[
M_1M \Im(H) = M_1 \Im(H_1) + \alpha_{eq1} M_2 \Im(H_2)
\]

\[
M_2M \Im(H) = M_1 \Im(H_1) + \alpha_{eq2} M_2 \Im(H_2)
\]

(where \( H_1 \) and \( H_2 \) are the value of \( H \) for disk and shaft respectively).

\[
M_1M = M_1 + \alpha_{eq1} M_2 \quad M_2M = M_1 + \alpha_{eq2} M_2
\]
in which $M_1$ and $M_2$ are the masses of the fluid displaced by the disc and shaft respectively.

From equation (17) and (18) we have

$\left(M_1M_4 + M_1M Re(H)\right) \frac{d^2x}{dt^2} - M_1M \omega \omega_H Im(H) \frac{dx}{dt} + K_{44} x$

$= M_1M_4 \omega^2 \cos \omega t$

$\left(M_2M_5 + M_2M Re(H)\right) \frac{d^2y}{dt^2} - M_2M \omega \omega_H Im(H) \frac{dy}{dt} + K_{55} y$

$= M_2M_5 \omega^2 \sin \omega t$

Equation (19) in dimensionless form can be written as

$\left[1 + M_1^* Re(H)\right] \frac{d^2\xi}{dt^2} - M_1^* \omega_1 \omega_H Im(H) \frac{d\xi}{dt} + \xi = \varepsilon^* (\omega_1)^3 \cos (\omega_1 t_1)$

$\xi = \omega_1 x/R_1, \eta = \omega_2 y/R_2, \omega_1 = \omega/\omega_0, \omega_2 = \omega/\omega_0, \varepsilon^* = \varepsilon/R_1$

$M_1^* = M_1/M_1M_4, M_2^* = M_2/M_2M_5, t_1 = \omega_0 t_1, t_2 = \omega_0 t_2, K_{44} = \sqrt{K_{44}/M_1M_4}, K_{55} = \sqrt{K_{55}/M_2M_5}$

$\delta^*_1 = \delta^*_4, \delta^*_2 = \delta^*_5$ are the maximum dimensionless amplitude in 44- and 55- direction respectively.

**Numerical Analysis and Discussion on Results**

Numerical analysis was carried out for non dimensional amplitude ratio ($\delta^*/\varepsilon^*$) of a mild steel shaft specimen with different crack depth ($\beta\beta$), crack width $(2b)$ and crack position ($\alpha\alpha=0.5$) by varying non dimensional frequency ratio $(\omega/\omega_0)$ using expression (23). All the numerical results have been plotted in Fig.3 to Fig.10. The mild steel shaft specimen considered for numerical analysis was of length $1.2\ m.$, cross-sectional radius $0.012\ m$ with a central disc of $1.96\ kg.$ mass, $0.04\ m.$ radius and $0.05\ m.$ width. Fig.3 to Fig.5 show the variation of amplitude ratio vs frequency ratio for the shaft specimen with different relative crack depths in two directions in a unique viscous medium $(v=2.3)$. In Fig.6 a comparison is made between dimensionless amplitudes, by varying the gap ratio, it is observed that as the gap ratio increases the amplitude also increases. In Fig.7 variation of natural frequency ratio vs relative crack depth is given. As the crack depth increases the natural frequency ratio decreases. From Fig.8 to Fig.10 exhaustive comparisons are made among dimensionless amplitude vs frequency ratio for various crack depths, in various viscosity media.

From equation (17) and (18) we have

$\frac{d^2x}{dt^2} + M_1M Re(H) \frac{dx}{dt} + K_{44} x + \frac{M_1M \omega \omega_H Im(H) \frac{dx}{dt}}{M_1M_4} = M_1M_4 \omega^2 \cos \omega t$

$\frac{d^2y}{dt^2} + M_2M Re(H) \frac{dy}{dt} + K_{55} y + \frac{M_2M \omega \omega_H Im(H) \frac{dy}{dt}}{M_2M_5} = M_2M_5 \omega^2 \sin \omega t$

Equation (19) in dimensionless form can be written as

$\frac{d^2\xi}{dt^2} - \omega_1 \omega_0 \omega_H \omega_1 \omega_2 \omega_0 \omega_0 \frac{d\xi}{dt} + \xi = \varepsilon^* (\omega_1)^3 \cos (\omega_1 t_1)$

$\xi = \omega_1 x/R_1, \eta = \omega_2 y/R_2, \omega_1 = \omega/\omega_0, \omega_2 = \omega/\omega_0, \varepsilon^* = \varepsilon/R_1$

$M_1^* = M_1/M_1M_4, M_2^* = M_2/M_2M_5, t_1 = \omega_0 t_1, t_2 = \omega_0 t_2, K_{44} = \sqrt{K_{44}/M_1M_4}, K_{55} = \sqrt{K_{55}/M_2M_5}$

$\delta^*_1 = \delta^*_4, \delta^*_2 = \delta^*_5$ are the maximum dimensionless amplitude in 44- and 55- direction respectively.

**Numerical Analysis and Discussion on Results**

Numerical analysis was carried out for non dimensional amplitude ratio ($\delta^*/\varepsilon^*$) of a mild steel shaft specimen with different crack depth ($\beta\beta$), crack width $(2b)$ and crack position ($\alpha\alpha=0.5$) by varying non dimensional frequency ratio $(\omega/\omega_0)$ using expression (23). All the numerical results have been plotted in Fig.3 to Fig.10. The mild steel shaft specimen considered for numerical analysis was of length $1.2\ m.$, cross-sectional radius $0.012\ m$ with a central disc of $1.96\ kg.$ mass, $0.04\ m.$ radius and $0.05\ m.$ width. Fig.3 to Fig.5 show the variation of amplitude ratio vs frequency ratio for the shaft specimen with different relative crack depths in two directions in a unique viscous medium $(v=2.3)$. In Fig.6 a comparison is made between dimensionless amplitudes, by varying the gap ratio, it is observed that as the gap ratio increases the amplitude also increases. In Fig.7 variation of natural frequency ratio vs relative crack depth is given. As the crack depth increases the natural frequency ratio decreases. From Fig.8 to Fig.10 exhaustive comparisons are made among dimensionless amplitude vs frequency ratio for various crack depths, in various viscosity media.

For $n=1$ and 2

$\delta^*_1 = \delta^*_4 = (\delta^*_1)^2 + (\delta^*_2 \cdot \delta^*_4)^2)^{0.5}$

$\delta^*_2 = \delta^*_5 = (\delta^*_2)^2 + (\delta^*_1 \cdot \delta^*_5)^2)^{0.5}$

where $\delta^*_4 = \sin (\omega_1^* t_2 - \phi_2)$, when $(\omega_1^* t_1 - \phi_1) = 0$

$\delta^*_5 = \sin (\omega_2^* t_2 - \phi_2)$, when $(\omega_2^* t_2 - \phi_2) = \pi/2$ (23)

$\delta^*_1 (= \delta^*_4)$ and $\delta^*_2 (= \delta^*_5)$ are the maximum dimensionless amplitude in 44- and 55- direction respectively.

![Fig. 3 Frequency ratio Vs. dimensionless amplitude ratio](image-url)
Fig. 4 Frequency ratio Vs. dimensionless amplitude ratio 
mild steel shaft \( R_l = 0.012 \text{m}, L = 1.2 \text{m} \), relative crack position (\( \alpha \alpha \)) = 0.5, relative crack depth \( (\beta \beta) = 0.34 \), 
gap ratio \( (q') = 4.50 \)

Fig. 5 Frequency ratio Vs. dimensionless amplitude ratio 
mild steel shaft \( R_l = 0.012 \text{m}, L = 1.2 \text{m} \), relative crack position (\( \alpha \alpha \)) = 0.5, relative crack depth \( (\beta \beta) = 0.44 \), 
gap ratio \( (q') = 4.50 \)

Fig. 6 Frequency ratio Vs. dimensionless amplitude ratio 
mild steel shaft \( R_l = 0.012 \text{m}, L = 1.2 \text{m} \), relative crack position (\( \alpha \alpha \)) = 0.5, relative crack depth \( (\beta \beta) = 0.34 \), 
gap ratio \( (q') = 4.50 \)

Fig. 7 Relative crack depth \( (\beta \beta) \) Vs. natural frequency ratio 
\( \left( \frac{\omega_0}{\omega_0 / \omega_0} \right) \), mild steel shaft \( R_l = 0.012 \text{m}, L = 1.2 \text{m} \), relative crack position (\( \alpha \alpha \)) = 0.5, gap ratio \( (q') = 12.0 \)

Fig. 8 Frequency ratio Vs. dimensionless amplitude ratio 
mild steel shaft \( R_l = 0.012 \text{m}, L = 1.2 \text{m} \), relative crack position (\( \alpha \alpha \)) = 0.5, relative crack depth \( (\beta \beta) = 0.2 \), 
gap ratio \( (q') = 4.50 \)

Fig. 9 Frequency ratio Vs. dimensionless amplitude ratio 
mild steel shaft \( R_l = 0.012 \text{m}, L = 1.2 \text{m} \), relative crack position (\( \alpha \alpha \)) = 0.5, relative crack depth \( (\beta \beta) = 0.3 \), 
gap ratio \( (q') = 4.50 \)
Conclusions

The investigation of vibrational behavior of a cracked rotor in a liquid medium was conducted. From the above analysis and discussions the following conclusions have been arrive at.

From Fig.3 to Fig.5 it is evident that, as the viscosity of the fluid medium increases the critical speed of the rotor shaft decreases due to increase in virtual mass (i.e. $|\text{mRe}(H)|$) and also the amplitude decreases due to increasing in damping factor (i.e. $|\text{m}_\text{c0 Re}(H)|$).

As the gap ratio increases, the amplitude of vibration increases, which is due to decrease in damping factor and virtual mass effect. The effect can be visualised from Fig.6.

Due to an increase in crack depth, the natural frequency decreases, this because of a decrease in stiffness of the shaft, and the effect is highlighted in Fig.7.

From Fig.8 to Fig.10 it is concluded that, due to presence of the crack, the critical speed decreases and as the stiffness of the cracked shaft along the crack (55-direction) is lower than the stiffness perpendicular to the crack (44-direction), the critical speed measured along the crack (55-direction) is lower than that measured in the direction perpendicular to the crack (44-direction).

Because of the crack the critical speed of the shaft decreases and due to that low critical speed the damping co-efficient ($|\text{m}_\text{c0 Re}(H)|$) increases for which, the maximum dimension-less amplitude of the rotating cracked shaft is lower than that of the uncracked shaft. The above mentioned effect can be visualised from Fig.8 to Fig.10.

Due to an increase in relative crack depth ($\beta$) of the rotor the critical speed decreases, which is observed from Fig.3 to Fig.10. Because of the decrease in critical speed the damping effect increases for which amplitude of vibration decreases subsequently, as shown in Fig.3 to Fig.10.

The above dynamic behavior can be utilized for the analysis of a cracked rotating shaft in a viscous medium and for fault detection of a rotor rotating in any viscous medium for condition monitoring.

References


Appendix

Analysis of Vibrational Characteristics of the Cracked Shaft

A simply supported shaft of length 'L' radius 'R', with a crack of depth 'at' at a distance 'Li' from one end is being considered (as shown in Fig.2). Taking \( \theta_1 (z,t) \) and \( \theta_2 (z,t) \) as deflection of torsional vibrations for the sections before and after the crack, \( Y_1 (z,t) \), \( Y_2 (z,t) \) are the deflection for bending vibrations and \( \xi_1 (z,t), \xi_2 (z,t) \) are the slope of the deflection curve for the same sections.

The normal function for the system in dimension-less form can be defined as:

\[
\begin{align*}
\bar{\theta}_1 (z) &= A_1 \cos (K_u z) + A_2 \sin (K_u z) \\
\bar{\theta}_2 (z) &= A_3 \cos (K_u z) + A_4 \sin (K_u z) \\
\bar{Y}_1 (z) &= a_1 \sin (\lambda_1 z) + b_1 \cos (\lambda_1 z) + c_1 \sinh (\lambda_2 z) + d_1 \cosh (\lambda_2 z) \\
\bar{Y}_2 (z) &= a_2 \sin (\lambda_1 z) + b_2 \cos (\lambda_1 z) + c_2 \sinh (\lambda_2 z) + d_2 \cosh (\lambda_2 z) \\
\bar{\xi}_1 (z) &= a_1 A_1 \cos (\lambda_1 z) - b_1 A_1 \sin (\lambda_1 z) + c_1 A_2 \cosh (\lambda_2 z) + d_1 A_2 \sinh (\lambda_2 z) \\
\bar{\xi}_2 (z) &= a_2 A_1 \cos (\lambda_1 z) - b_2 A_1 \sin (\lambda_1 z) + c_2 A_2 \cosh (\lambda_2 z) + d_2 A_2 \sinh (\lambda_2 z)
\end{align*}
\]

where

\[
\begin{align*}
\bar{z} &= \frac{z}{L}, \bar{\theta} &= \frac{\theta_1}{L}, \bar{Y}_i = \frac{Y_i}{L}, \bar{\xi}_i = \frac{\xi_i}{L} \text{ for } i=1,2 \\
\bar{K} &= \frac{\omega L}{C_y}, C_y = \left( \frac{G L}{\rho} \right)^2, \mu = A_1 \rho_1 \\
\lambda_1^2 &= \frac{\mu^2 L^4 \omega^2}{A_1 E} \left( 1 + \frac{E}{kk G} \right), \psi^2 = \frac{\mu^2 L^4 \omega^2}{E I} \left( \frac{1 - \rho_1}{\rho_1} \right) \\
\lambda_2^2 &= \frac{(\psi^2 + \lambda_1^2/4 + \lambda_2^2/2)^{0.5}}{2}, \lambda_2 = \frac{(\psi^2 + \lambda_1^2/4 - \lambda_2^2/2)^{0.5}}{2} \\
A_1 &= (\lambda_1 - \lambda_2)/\lambda_1, A_2 = (\lambda_2 + \lambda_3)/\lambda_2, A_3 = \omega/(kk GA_1/(\mu \mu L^3))^{0.5}
\end{align*}
\]

A_1 = Shaft Cross-section
\( \rho_1 = \) Mass density of the material
\( E = \) Young's Modulus of Elasticity
\( G = \) Shear Modulus
\( \nu = \) Poisson's Ratio
\( \mu = \) Modulus of rigidity

The boundary conditions of the simply supported shaft in consideration are:

\[
\begin{align*}
Y_1 (0) &= 0 \\
E I \bar{\xi}_1 (0) &= 0
\end{align*}
\]
\[ Y_2 (L) = 0 \quad \text{(iv)} \]
\[ E \xi_1' (L) = 0 \quad \text{(v)} \]
\[ E \xi_1' (L_1) = E \xi_2' (L_1) \quad \text{(vi)} \]
\[ \kappa K A_1 (y_1' (L_1) - \xi_1 (L_1)) = K K A_1 (Y_2' (L_1) - \xi_2 (L_1)) \quad \text{(vii)} \]
\[ \kappa K A_1 (y_1' (L_1) - \xi_1 (L_1)) = K K A_2 (y_2' (L_1) - y_1 (L_1)) \]
\[ + \kappa K A_6 (\Omega_2 (L_1) - \Omega_1 (L_1)) \quad \text{(viii)} \]
\[ E I \xi_1' (L_1) = K K A_4 (\xi_2 (L_1) - \xi_1 (L_1)) \quad \text{(ix)} \]
\[ G I_p \Omega_1' (0) = 0 \quad \text{(x)} \]
\[ G I_p \Omega_2' (L) = 0 \quad \text{(xi)} \]
\[ G I_p \Omega_1' (L_1) = G I_p \Omega_2' (L_1) \quad \text{(xii)} \]
\[ G I_p \Omega_1' (L_1) = K K A_5 (y_2 (L_1) - y_1 (L_1)) \]
\[ + \kappa K A_6 (\Omega_2 (L_1) - \Omega_1 (L_1)) \quad \text{(xiii)} \]

\[ J(a) = \text{Strain energy density function} \]
\[ P_i, i=1 \text{ to } 6 \]
\[ \text{Axial force (i=i)} \]
\[ \text{Shear force (i=2,3)} \]
\[ \text{Bending moment (i=4,5)} \]
\[ \text{Torsional moment (i=6)} \]

2b = width of the crack, a1 = Depth of the crack, a2 = h1/2 (h1, a1, a2, a3 are shown in Fig.2).

Applying boundary conditions in characteristic equation, the fundamental critical speed in \( \omega^{55} \) (55-dirn) and \( \omega^{44} \) (44-dirn) are found out.

The global stiffness matrix \([K_g]\) can be written as:

\[ [K_g] = \begin{bmatrix} K_{55} & K_{54} & K_{51} \\ K_{45} & K_{44} & K_{41} \\ K_{15} & K_{14} & K_{11} \end{bmatrix} = [(G_1) [C_{el}] [G_2]]^{-1} \]

where \([G_1]=\text{diag} [L/2, L/2, L/2, 1], [G_2] = [L/4, L/4, 1]\)

\[ [C_{el}] = \begin{bmatrix} \frac{L^3}{48} & \frac{L^3}{48} & \frac{L}{EA_1} \\ \frac{L^3}{48} & \frac{L^3}{48} & \frac{L}{EA_1} \\ \frac{L}{EA_1} & \frac{L}{EA_1} & \frac{L}{EA_1} \end{bmatrix}, \]

\[ [G_1] = \begin{bmatrix} C_{55}/R_1 & C_{54}/R_1 & C_{51} \\ C_{45}/R_1 & C_{44}/R_1 & C_{41}/R_1 \\ C_{15}/R_1 & C_{14}/R_1 & C_{11}/R_1 \end{bmatrix} \]

where \(F_0 = A_1 E (1 - v_1^2), \text{diag} [...] = \text{diagonal matrix.} \)
Dynamic deflection of beam due to moving mass

A.K. Behera* and Dayal R. Parhi*

This paper presents the analysis of dynamic deflection of a cantilever beam subjected to a moving mass. The transport theorem of continuum mechanics is used for formulation of the equation for the system. Runge-Kutta method has been used to solve the differential equation for theoretical analyses. Experimental investigations are also carried out for comparison with the corresponding numerical results. The results are found to be in good agreement.

Theoretical as well as experimental investigation on the behaviour of beams subjected to moving masses are quite important because of premature failure of structures, like bridges and industrial structures. No systematic work has been reported for evaluation of dynamic response of structures carrying moving mass which still remains as a problem of practical significance. In early part of the twentieth century, Jeffcott* managed to evaluate the dynamic response of structures with moving mass for simple cases, assuming the mass as moving force.

Investigations of Steele2, Florence3, Smith4 and Kenney5 were mainly focussed on the prediction of response characteristics of a beam subjected to moving forces so as to eliminate inertia effects of the moving body.

Fryba6 analysed the vibration characteristics of a beam subjected to moving force of random in nature. Kurihara and Simogo7 have analysed the stability criteria of a simply supported beam to randomly spaced moving loads, and have also analysed vibration characteristics of an elastic beam subjected to discrete moving loads. Saigal8 has developed theoretical expressions for beam type structures with the help of Stanisic's theory9, which has a higher degree of practical significance. Akin et al.10 worked on similar problems considering finite beam with moving load.

Despite the ever increasing number of research publications on dynamic response of structures with moving mass, neither the effect of the mass velocity nor the continuum effect have been properly highlighted.

In this paper, a systematic approach has been developed, taking moving mass (inertia) and continuum effect, to account for a wide range of mass velocities and for various magnitudes of the moving mass. Experiments were also conducted by changing various parameters and results were compared with the numerical values evaluated from theoretical analysis.

EQUATION OF MOTION

The equation of motion of an uniform beam of mass $m$ (per unit length) subjected to a moving mass $M$, can be written neglecting damping as:

$$EI \frac{d^4y(x,t)}{dx^4} + m \frac{d^2y(x,t)}{dt^2} = P(x,t)$$  \hspace{1cm} (1)

The external force $P(x,t)$ can be taken as:

$$P(x,t) = F(t) \delta(x-\nu t), \hspace{1cm} \text{and}$$

$$F(t) = MG - M \frac{d^2}{dt^2} + \nu \frac{d^2}{dt^2} \gamma(\beta, \nu) \mid \nu = \nu t$$  \hspace{1cm} (2)

From equation (1), (2) and (3), we get:

$$EI \frac{d^4y(x,t)}{dx^4} + \rho A \frac{d^2y(x,t)}{dt^2} = \left(Mg - M \left(\frac{d^2}{dt^2} + \nu \frac{d^2}{dt^2}\right) \gamma(\beta, \nu) \right) \delta(x-\beta)$$  \hspace{1cm} (4)

The solution of Eq.(4) is assumed in a series form as:

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x) T_n(t)$$  \hspace{1cm} (5)

where $Y_n(x)$ is the eigen-function of the beam (without $M$) and is obtained from:

$$\gamma_n(x) = \gamma_n \gamma_n(x) = 0$$  \hspace{1cm} (6)

* Department of Mechanical Engineering, Regional Engineering College, Rourkela, Orissa, India 769008
where \( \gamma_n^A = \rho A \frac{\omega_n^2}{E} l \)

The general solution of Eq. (6) can be written as:

\[
Y_n(x) = a \sin(\gamma_n x) + b \cos(\gamma_n x) + c \sinh(\gamma_n x) + d \cosh(\gamma_n x)
\]

where \( a, b, c, \) and \( d \) are the constants.

From orthogonality principle and from the orthogonal properties of the function \( Y_n(\cdot) \), right side of Eq. (9):

\[
\sum_{n=1}^{\infty} \int Y_n(x) \delta(x - \beta) \, dx = \sum_{n=1}^{\infty} Y_n(x) \delta(x - \beta)
\]

Substituting Eq. (5) into Eq. (8) and multiplying both sides by \( Y_p(x) \) and then integrating, we obtain:

\[
\int Y_p(x) \delta(x - \beta) \, dx = \sum_{n=1}^{\infty} \int Y_p(x) Y_n(x) \delta(x - \beta) \, dx
\]

Using,

\[
Mg \int_{\alpha}^{\beta} Y_p(x) \delta(x - \beta) \, dx = Mg Y_p(\beta)
\]

We get

\[
M \sum_{n=1}^{\infty} \int Y_p(x) Y_n(x) \delta(x - \beta) \, dx = \sum_{n=1}^{\infty} \int Y_p(x) Y_n(x) \delta(x - \beta) \, dx
\]

\[
= M \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) \delta(x - \beta) \, dx
\]

\[
= M \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) \delta(x - \beta) \, dx
\]

\[
= M \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) \, dx
\]

From orthogonality principle and from the orthogonal properties of the function \( Y_n(\cdot) \), right side of Eq. (9):

\[
\sum_{n=1}^{\infty} \int Y_p(x) Y_n(x) S_n(t) \, dx = \sum_{n=1}^{\infty} S_n(t) \int Y_p(x) Y_n(x) \, dx
\]

Using Eqs. (10), (11) and (12), Eq. (9) reduces to:

\[
S_p(t) = Mg \sum_{n=1}^{\infty} \left[ \delta(x - \beta) \frac{\partial}{\partial t} + \frac{\partial}{\partial \beta} \right] Y_n(\beta) \delta(x - \beta) \, dx
\]

\[
= S_p(t) V_p
\]

From Eqs. (4) and (8), we have

\[
\frac{d^2 Y(x,t)}{dt^2} + \rho A \frac{\partial^2 Y(x,t)}{\partial \beta^2} = \sum_{n=1}^{\infty} Y_n(x) S_n(t)
\]

From Eqs. (13), (14) and (5), we obtain

\[
\frac{d^2 Y(x,t)}{dt^2} + \rho A \frac{\partial^2 Y(x,t)}{\partial \beta^2} = \sum_{n=1}^{\infty} Y_n(x) S_n(t)
\]

\[
= \sum_{n=1}^{\infty} Y_n(x) \frac{Mg}{V} \left[ g - \sum_{q=1}^{\infty} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \beta} \right)^2 Y_q(\beta) T_q(t) \right] Y_n(\beta)
\]

By substituting the value of \( Y_n(\cdot) \) from Eq. (6) into Eq. (15), we obtain

\[
\sum_{n=1}^{\infty} \int Y_p(x) \left[ \frac{d^2 Y(x,t)}{dt^2} + \rho A T_n(t) \right] Y_n(\beta)
\]

\[
= 0
\]
Eq. (17) is a differential equation of 2nd order, which is solved using Runge-Kutta method.

NUMERICAL ANALYSIS

The numerical results for the displacement at the points, \( x = 'end point' \); \( x = 'v' t' \) of the mild steel cantilever beam specimen (1.2 x 0.05 x 0.006m) for different positions of the moving mass \( (M = 1.0 \text{ kg}; 2.0 \text{ kg}) \) are calculated. These results are presented in Figures 2 to 7. In Figures 2 to 4, displacement at the points, \( x = 'end point' \); \( x = 'v' t' \) of the beam have been plotted with time, for various velocities (20 to 150 Kmph) of the moving mass \((M = 1.0 \text{ kg})\). In Figures 5 to 7, similar results have been plotted for the other moving mass \((M = 2.0 \text{ kg})\).

For solving Eq. (5), the values of \( T_n(t) \) are obtained at required position from Eq. (17), using Runge-Kutta technique.

Numerical calculations are carried out in a computer using first six modes of vibration amplitudes in Eq. (5) for convergence of the result with accuracy of order \( 10^{-4} \) units.

From Figures 2 to 7, it is observed that as the velocity of the moving mass increases, the deflection at the end point of the beam decreases. This is because at higher velocity of the moving mass, lower modes are not exited, which mainly contribute for larger dynamic displacement of the beam. As the lower modes of vibration contribute, large amplitude as compared to that of higher modes, dynamic displacement of the beam decreases.

It is noticed from Figures 3, 4, 6 and 7 that at the free end, dynamic deflection of the beam decreases sharply with
It is also found that, as the mass of the moving body increases, the end deflection of the beam also increases. This is due to increase in inertia effect of the moving mass.

The points discussed above can also be visualised from experimental results shown in Figures 8 to 13.

time and again increases at higher mass velocities because of higher modes of vibration are more prominent than that of 1st mode. However, at $x = v \cdot t$, as the moving mass is on the point where the dynamic deflection is to be recorded, the dynamic deflection mainly depends upon the weight of the moving mass, whereas at the end point of the beam, $(x = L)$, the dynamic deflection mainly depends on the modal vibration of the team. Therefore, dynamic deflection exhibits sharp changes in magnitude as well as direction at the free end.

It is also found that, as the mass of the moving body increases, the end deflection of the beam also increases. This is due to increase in inertia effect of the moving mass.

The points discussed above can also be visualised from experimental results shown in Figures 8 to 13.

**EXPERIMENTAL SET-UP**

The schematic diagram of the experimental set-up used for the experiments is shown in Figure 14. A number of tests were conducted on mild steel beam specimens
(1.2 x 0.05 x 0.006m) with two different moving masses, (1.2 and 2.0kg) as mentioned in numerical analysis, for determining dynamic deflection at the free end of the cantilever beam specimen.

The experimental results of free end deflection for various positions of the moving mass were recorded by positioning the LVDT at the free end. It was so arranged in the set-up that it could be engaged or disengaged as required. The output of the LVDT is fed to the displacement indicator in order to measure the highest instantaneous displacement of the free end of the specimen using the peak-hold mode, as the moving mass reaches a predetermined location on the cantilever beam specimen.

While conducting the experiments, sufficient precautions have been taken, in positioning the LVDT and controlling the variac, for obtaining the desired velocity of the moving mass in order to achieve the best possible accuracy in all measurements.

The above results for different velocities of the moving mass are plotted in Figures 8 to 13. Corresponding numerical results are also presented in the same graph for comparison. It is quite evident from the graphs that both experimental and numerical results show a very good agreement.

CONCLUSIONS
Based on the above discussions, the concluding remarks for the present investigation are depicted as follows:

Velocity of the moving mass has got significant role in deciding the deflection of beam structures. With the increase in velocity of the moving mass, the dynamic deflection of structures show a decreasing trend. This is because the lower modes are not excited at higher velocity.

With the increase in weight of the moving mass, the dynamic deflection of the structure increases.

Further study can be extended using these information for finding out dynamic deflection of complicated structures used in actual practice with moving mass system. Once the maximum deflection in such structures are predicted, it will be possible to evaluate the maximum stress in the member of the structure, which will help in its design.

REFERENCES
4. Smith, C.E., "Motion of a Stretched String Carrying a...


APPENDIX

The Dirac-delta function \( \delta (x - \beta) \) has the following properties:

\[
\delta (x - \beta) = 0 \quad \text{for} \quad x \neq \beta
\]

= greater than any assumed value for \( x = \beta \)

\[
\int_{-\infty}^{\infty} f(x) \delta (x - \beta) \, dx = f(\beta) \quad 0 < \beta < \infty
\]

Then for a beam of length \( L \)

\[
\int_{0}^{L} f(x) \delta (x - \beta) \, dx = f(\beta) \quad 0 < \beta < L
\]

and

\[
\int_{0}^{L} f(x) \delta (x - \beta) \, dx = 0 \quad \beta < 0, \beta > L
\]
The Study of Virtual Mass and Damping Effect on a Rotating Shaft in Viscous Medium

D R Parhi, Associate Member
Dr A K Behera, Fellow

This paper outlines a complete analysis of virtual mass and damping effects on the rotating shaft in viscous fluid. The system taken for analysis is a simply supported shaft rotating in finite fluid region. With the help of Navier-Stokes equation, the fluid forces acting on the submerged rotating shaft is found out. The virtual mass effect and the damping effect are evaluated from the fluid force. It is observed that both virtual mass and damping effect are responsible in changing the amplitude of vibration. While the virtual mass effect is mainly responsible for shifting the critical speed of the system.

Keywords: Virtual mass and damping effect; Viscous medium; Rotating shaft

NOTATION

- \( D \) : diameter of the shaft
- \( \delta \) : whirling radius of the shaft
- \( E \) : modules of elasticity of shaft material
- \( \varepsilon \) : eccentricity
- \( F_x, F_y \) : fluid forces on rotor in x and y directions respectively
- \( I \) : section moment of inertia of the shaft
- \( L \) : total length of the shaft
- \( m \) : mass of the fluid displaced by the shaft per unit length (\( \rho \pi R^2 \))
- \( m_s \) : mass of the shaft per unit length
- \( m^* \) : dimensionless parameter (= \( m/m_s \))
- \( \eta_1 \) : gap ratio \([ (R_s - R_i)/R_i ]\)
- \( R_i \) : radius of the shaft
- \( R_s \) : radius of the cylinder
- \( t \) : time
- \( \alpha \) : \( k R_i \)
- \( \beta \) : \( k R_1 \)
- \( \mu \) : coefficient of viscosity
- \( \nu \) : coefficient of kinematic viscosity
- \( \rho \) : fluid density
- \( \omega_0 \) : fundamental natural frequency of shaft = \( \sqrt{2} \frac{EI/(m_s L^4)}{m} \)

INTRODUCTION

Due to the development of high speed machinery, robots, and aerospace structures, the research on rotor dynamics is in limelight. If there is external viscous medium, the critical speed and amplitude of vibration changes drastically. And if the rotor is rotating in a viscous fluid container then the container radius has a grip on amplitude of vibration.

The study of free vibrations for a stationary Timoshenko beam can be traced back to the middle of this century when Kruszezkwie\( \ddot{z} \) obtained the frequency equations for the fixed-free and the free-free beams. Anderson\( ^7 \) and Dolph\( ^3 \) presented general solutions for the simple-supported boundary conditions. An almost complete theoretical treatment of the problem was published by Trail-Nash and Collar\( ^7 \), giving the frequency equations and mode shapes for all six types of boundary conditions, namely, simply-supported or hinged-hinged, hinged-free, free-free, fixed-fixed, fixed-free, and fixed-hinged. However, they only presented results for half the mode shapes as those associated with the bending angle were not derived.

Matsusita and Ida\( ^5 \) in their paper used quasi-modal-transformation method for analyzing rotor vibration due to harmonic excitation, in particular, to excitation with frequency multiple of rotational speed, with an aim of computer aided designing for unbalance vibration.

Achenbach and Qu\( ^6 \) investigated about the forced vibration of a submerged beam of circular cross-section by using mathematical methods. But they have not shown the effect of damping and virtual mass separately. Zu and Han\( ^8 \) carried out a complete free vibration analysis of a spinning, finite Timoshenko beam with general boundary conditions. Subsequently Zu and Han\( ^8 \) carried out the dynamic response of a spinning Timoshenko beam with general boundary conditions and subjected to a moving load. Recently, Segalman\( ^6 \), et al used quadratic component method to analyse the dynamics of rotating flexible structures.

So far no systematic approach has been followed to find out the virtual mass effect, damping effect and gap ratio (container radius) effect on the rotor in viscous fluid. In the present investigation, factors governing the virtual mass, damping and gap ratio are given in mathematical terms, which makes the system behaviour understandable in a simple way. This analysis will also help in predicting the vibrational behaviour of rotors in viscous fluid and also for condition monitoring.

GOVERNING EQUATIONS FOR FLUID MOTION

A shaft of cross-sectional radius \( R_i \) is rotating with a speed \( \omega \) having whirling speed \( \Omega \) with \( \delta \) as the radius, is shown in Fig 1(b).

The Navier-Stokes equation in polar coordinate is generally expressed as;
\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nabla \cdot \left( \frac{\partial u}{\partial t} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} - \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right) \\
\frac{\partial v}{\partial t} &= \frac{1}{\rho} \frac{\partial p}{\partial \theta} + \nabla \cdot \left( \frac{\partial v}{\partial t} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{1}{r^2} \frac{\partial v}{\partial \theta} + \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right)
\end{align*}
\]  
(1)

In the above equation  \( u \) and  \( v \) denote the flow velocity in radial and tangential directions respectively and  \( p \) is the fluid pressure. With the help of stream function  \( \psi (r, \theta, t) \) the above equations can be written as;

\[
\nabla^2 \psi = 0, \quad \nabla^2 \psi - \left( 1/\sqrt{\nu} \right) \frac{\partial^2 \psi}{\partial \theta^2} = 0
\]  
(2)

The solution of equation (2) is expressed as;

\[
\psi = \psi_1 + \psi_2
\]

where  \( \psi_1 \) and  \( \psi_2 \) are solutions of equation (3)

The radial and tangential components of flow velocity at point  \( A \) (Fig 1) are;

\[
\begin{align*}
\begin{aligned}
\frac{\partial \psi}{\partial r} &= R_0 \sin (\alpha) - \delta \cos \left( \frac{\psi}{R_0} \right) \\
\frac{\partial \psi}{\partial r} &= R_0 \cos \left( \frac{\psi}{R_0} \right)
\end{aligned}
\end{align*}
\]  
(4)

The non-stationary components of flow velocities can be written as;

\[
\begin{align*}
\begin{aligned}
\psi_1 (r, \theta, t) &= F_1 (r) e^{i\left( \omega t - \theta \right)} \\
\psi_2 (r, \theta, t) &= F_2 (r) e^{i\left( \omega t - \theta \right)}
\end{aligned}
\end{align*}
\]  
(8)

From equations (8) and (3), one obtains;

\[
\begin{align*}
\frac{d^2 F_1}{dr^2} + \frac{1}{r} \frac{d F_1}{dr} - \frac{1}{r^2} F_1 &= 0
\end{align*}
\]  
(9)

The equation (9) is Euler’s equation and equation (10) is Bessel’s equation. The solution to the above equations can be written as;

\[
F_1 (r) = \delta \cos \left( \frac{\psi}{R_0} \right) + \delta \sin \left( \frac{\psi}{R_0} \right)
\]

\[
F_2 (r) = \delta \sin \left( \frac{\psi}{R_0} \right)
\]  
(10)

Where  \( A, B, C \) and  \( D \) are arbitrary constants and  \( \psi \) and  \( \psi_1 \) are modified Bessel’s function of first and second kind respectively.

The non-stationary components of flow velocities can be written as;

\[
\begin{align*}
\psi_1 (r, \theta, t) &= F_1 (r) e^{i\left( \omega t - \theta \right)} \\
\psi_2 (r, \theta, t) &= F_2 (r) e^{i\left( \omega t - \theta \right)}
\end{align*}
\]  
(11)

FORMULATION FOR FLUID FORCES

Substituting the flow velocities given by equation (11) into equation (1), the non-stationary component of pressure  \( p \) can be written as;

\[
P = \int \frac{\partial p}{\partial \theta} d\theta = \delta \omega^2 \left[ A - B - C - D \right] e^{i\left( \omega t - \theta \right)}
\]  
(13)

Normal stress  \( \tau_{rr} \) and tangential stress  \( \tau_{rt} \) due to flow can be written as;

\[
\begin{align*}
\tau_{rr} &= -p + 2\mu \frac{\partial u}{\partial r} \\
\tau_{rt} &= \mu \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right)
\end{align*}
\]  
(14)

Fluid forces acting on the surface \( (r = R_1) \) per unit length of the shaft in the  \( x \) and  \( y \) directions are written as;

\[
F_x = \int \left( \tau_{rr} \cos \theta - \tau_{rt} \sin \theta \right) R_1 d\theta
\]

\[
F_y = \int \left( \tau_{rr} \sin \theta + \tau_{rt} \cos \theta \right) R_1 d\theta
\]

(15)

The coordinates of the centre of the shaft [as shown in Fig 1(b)] is  \( x = \delta \cos \omega t, y = \delta \sin \omega t \)
Now, \( F_x = -m \text{Re}(H) \frac{d^2 x_2}{dt^2} + m\omega \text{Im}(H) \frac{dx}{dt} \)
\( F_y = -m \text{Re}(H) \frac{d^2 y_2}{dt^2} + m\omega \text{Im}(H) \frac{dy}{dt} \)

(17)

In equation (16) \( m \text{Re}(H) \) denotes the virtual or added mass of the fluid relating to the inertia force of the shaft and \(-m\omega \text{Im}(H)\) denotes the viscous damping coefficient.

**EXPRESSION FOR ROTOR AMPLITUDE**

Here a simply supported rotating shaft immersed in finite fluid region is considered.

The equations of motion for the shaft having uniformly distributed mass are:

\[
\begin{align*}
\frac{d}{dt} \left( m_s \frac{d^2 x}{dr^2} + m \text{Re}(H) \frac{dx}{dr} \right) + m \omega^2 \text{Im}(H) x &= F_x \\
\frac{d}{dt} \left( m_s \frac{d^2 y}{dr^2} + m \text{Re}(H) \frac{dy}{dr} \right) + m \omega^2 \text{Im}(H) y &= F_y
\end{align*}
\]

(18)

where \( m_s = \text{mass of the shaft per unit length} \)
\( EI = \text{bending stiffness of the shaft} \)

From equations (17) and (18), one gets;

\[
\begin{align*}
\frac{d}{dr} \left( m_s + m \text{Re}(H) \right) \frac{d^2 x}{dr^2} - m \omega \text{Im}(H) \frac{dx}{dr} &= F_x \\
\frac{d}{dr} \left( m_s + m \text{Re}(H) \right) \frac{d^2 y}{dr^2} - m \omega \text{Im}(H) \frac{dy}{dr} &= F_y
\end{align*}
\]

(19)

Introducing dimensionless quantities \( \Xi = x/R, \eta = y/R, \zeta = z/R_i \)
\( e^* = \epsilon/R, L^* = L/R, m^* = m/m_s, \omega^* = \omega/\omega_0, \tau = \omega t \)

\( \omega_0 \) is the fundamental natural frequency of shaft = \( \pi^2 (EI/(m_s L_i^2))^{1/4} \)

Equation (19) reduces to;

\[
\begin{align*}
\left[ 1 + m^* \text{Re}(H) \right] \frac{d^2 \Xi}{dt^2} - m^* \omega^* \text{Im}(H) \frac{d\Xi}{dt} + \frac{L^* \omega^* \text{Im}(H)}{\pi^2} \frac{d^2 \Xi}{dt^2} &= e^* (\omega^*)^2 \cos (\omega^* \tau) \\
\left[ 1 + m^* \text{Re}(H) \right] \frac{d^2 \eta}{dt^2} - m^* \omega^* \text{Im}(H) \frac{d\eta}{dt} + \frac{L^* \omega^* \text{Im}(H)}{\pi^2} \frac{d^2 \eta}{dt^2} &= e^* (\omega^*)^2 \sin (\omega^* \tau)
\end{align*}
\]

(20)

Applying the Fourier transform to both sides of equation (20), one obtains,

\[
X_\nu (\tau) + C_{\nu} X_\nu + K_{\nu} X_\nu = A_\nu \cos (\omega^* \tau)
\]

(21)

where,

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The details of the system chosen for the numerical analysis are given in Figs [1(a) and 1(b)].

Fig 1(a) shows the rotor in finite fluid region, whereas Fig 1(b) shows the whirling motion of the rotor during rotating condition.

The results obtained from numerical analysis are plotted as shown in Figs 2 to 6, for various parameters (e.g., frequency ratio against dimension amplitude ratio, virtual mass effect, damping effect and gap ratio).

In Figs 2 and 3, the graphs are plotted between dimensionless amplitude ratio and frequency ratio. From these two figures, it is observed that as the viscosity of the external fluid increases, there is a shift in critical speed and decrease in amplitude of vibration.

Fig 1(a) Rotor in a finite liquid container

Fig 1(b) View at whirling condition of the rotor

(iii) Radius of the shaft, $R_1$, 0.011m
(iv) Damping coefficient, $v$, varies as 2.3 stokes, 0.427 stokes, and 0.0633 stokes
(v) Equivalent mass of fluid displaced/corresponding mass of the shaft, $m^*$, varies as 0.158, 0.1534 and 0.144
(vi) Gap ratio, $q_1 \left( \frac{(R_2 - R_1)}{R_1} \right)$, varies (2.0, 3.0, 4.0, 8.0 and 16.0)
(vii) radius of the viscous container, $R_2$
The effect of gap ratio is highlighted (Fig 4). It is observed with the increase in cylinder (fluid-filled) radius the corresponding dimensionless amplitude increases.

CONCLUSIONS

The vibration characteristics of simply supported spinning shaft in viscous fluid and the effect of damping, virtual mass and gap ratio are taken into consideration while concluding the results.

Because of increase in coefficient of viscosity, the amplitude of vibration decreases. Also critical speed shifts towards left. As the fluid-filled container radius (gap ratio) increases the amplitude of vibration increases because of decrease in virtual mass effect $1 - m \text{Im}(\text{Re}(H))$ and the damping effect $\text{Im}(\text{Re}(H))$.

The effect of virtual mass $\text{Im}(\text{Re}(H))$ shifts the critical speed towards left and also decreases the amplitude of vibration. But the effect of damping $1 - m \text{Im}(\text{Re}(H))$ reduces the amplitude of vibration.

The investigation carried out currently, can be used for vibration analysis of high speed shaft in viscous fluid, analysis or rotating shaft in high speed centrifuges, condition monitoring of dynamic rotor system.

REFERENCES

In this paper the Runge-Kutta method has been used to solve the differential equations involved in the analysis of dynamic deflection of a cantilever beam carrying a moving mass. Experimental results are compared with the corresponding numerical ones based on the above theory. A very-good agreement between them confirms the authenticity of the theory developed.

Nomenclature

A = Beam cross sectional area
B = Modulus of elasticity of beam material
F(t) = Load due to moving mass 'M'
g = Acceleration due to gravity
I = Section moment of inertia of the beam
L = Total length of the beam
M = Lumped mass of the moving body
m = Mass per unit length of the beam
n = Integer variable (varies from 1 to ∞)
P(x,t) = External load on the beam due to moving mass 'M'
Q(t) = General loading
q = Integer variable (varies from 1 to ∞)
t = Time
v = Velocity of the moving mass
\( Y_n = \int Y_n(X) Y_n(X) \, dx \)
x = The distance of the point where deflection is desired
Y_n = Eigen functions of the beam
y = Transverse dynamic deflection of the beam
\( \beta_n = v \, \gamma \)
\( \beta_n = \) Eigen-values
\( \delta = \) Dirac delta function
\( \rho = \) Mass density of the beam
\( \omega_n = \) Natural circular frequency of the beam 'nth mode'.
\( T_n(t) = \frac{\partial^2 T_n(t)}{\partial t^2} \)
\( = \frac{\partial}{\partial x} \)

Introduction

For several decades, engineers are investigating on the potential hazard produced due to moving masses on structures. The dynamic response of structures carrying moving masses is a problem of widespread practical significance. In the early part of the twentieth century, engineers such as Jeffcot, H [1] managed to calculate the dynamic response of structures with a moving mass for the simplest cases. However, their results were quite impractical for use in actual practice.

The investigations of Steele [2], Florence [3], Smith [4] and Kenney [5] were focused on determining response characteristics of a beam subjected to moving forces and not moving masses. Saigal, S [6] has developed expressions for the beam structures by the help of Stanisic's theory [7], which has got a higher degree of practical significance. Later Akin et al. [8] analysed such problems for the finite beam with a moving load using differential equations.

Despite the ever increasing number of research publications on dynamic response of structures with moving mass, the theory has not been verified with the experiments. The moving mass model has the advantage that the assumptions in developing the theory of moving force model can be avoided [8].

In order to remove such anomalies, a reliable theoretical method has been adopted in the present investigation in conjunction with experimental verification.
Equation of Motion

The equation of motion of a uniform beam of mass 'm' (mass per unit length) subjected to a moving mass 'M', can be written by, neglecting damping as:

$$ E I \frac{d^4 y(x,t)}{dx^4} + m \frac{d^2 y(x,t)}{dt^2} = P(x,t) $$

(1)

The external force $P(x,t)$ can be taken as

$$ P(x,t) = F(t) \delta(x-vt) $$

(2)

$$ F(t) = Mg - M \left( \frac{d}{dt} + \nu \frac{d}{dp} \right)^2 y(x,t) \text{[where, } \nu = vt \text{] } $$

(3)

where $F(t)$ is the force due to moving mass at that instant and $\delta$ is the Dirac-delta function.

The Dirac-delta function $\delta(x-x_0)$ has the following properties

$$ \delta(x-x_0) = 0 \text{ for all } x \neq x_0 $$

$$ = \text{greater than any assumed value for } x = x_0 $$

(4)

Then for a beam of length 'L'

$$ \int f(x) \delta(x-x_0) \, dx = f(x_0) \text{ if } 0 < x_0 < L $$

(4a)

and

$$ \int f(x) \delta(x-x_0) \, dx = 0 \text{ for } x_0 < 0 \text{ or } x_0 > L $$

(4b)

From equations (1), (2) and (3) we get

$$ E I \frac{d^4 y(x,t)}{dx^4} + \rho A \frac{d^2 y(x,t)}{dt^2} = \left[ Mg - M \left( \frac{d}{dt} + \nu \frac{d}{dp} \right)^2 \right] \delta(x-x) $$

(5)

Where $\beta = vt$, $m = \rho A = \text{mass per unit length of the beam}$ and $v = \text{velocity of the moving mass.}$

$t = \text{Time taken by the moving mass from the fixed end to a point on the beam, as shown in Fig. 1.}$

Assuming the solution of equation (5) in a series form i.e.

$$ y(x,t) = \sum_{n=1}^{\infty} Y_n(x) T_n(t) $$

(6)

Here $Y_n(x)$ is the eigen-function of the beam (without M) with the same boundary condition at nth mode.

To find out $Y_n(x)$ the equation will be

$$ Y_n''(x) - \beta_n^2 Y_n(x) = 0 $$

(7)

Where $\beta_n^2 = \rho A \frac{\omega_n^2}{EI}$

$\omega_n$ = 1,2,3.... are the natural frequencies of the beam.

The general solution of equation (7) can be written as

$$ Y_n(x) = a \sin(\beta_n x) + b \cos(\beta_n x) + c \sinh(\beta_n x) + d \cosh(\beta_n x) $$

(8)

Where $a,b,c$ and $d$ are the constant co-efficients, depending upon the boundary conditions.

$T_n(t)$ are the function of time are to be calculated.
Re-writing the right-hand side of equation (5) as
\[
\left[ M g - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 \right] y(\beta, t) = \sum_{n=1}^{\infty} Y_n(\beta) S_n(t)
\]

(9)

Now substituting equation (6) into equation (9), equation (9) transforms to
\[
\left[ M g - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 \sum_{n=1}^{\infty} Y_n(\beta) T_n(t) \right] \delta(\beta - \beta') = \sum_{n=1}^{\infty} Y_n(\beta) S_n(t)
\]

(10)

multiplying both side of equation (10) by \( Y_p(x) \) and integrating over the beam length we get
\[
\int_0^L Mg Y_p(x) \delta(x - \beta') \, dx - M \sum_{n=1}^{\infty} \int_0^L Y_p(x) \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(x) T_n(t) \, dx = \sum_{n=1}^{\infty} \int_0^L Y_p(x) Y_n(x) S_n(t) \, dx
\]

(11)

From equation 4(a) and 4(b)
\[
\int_0^L Y_p(x) \delta(x - \beta) \, dx = Mg Y_p(\beta),
\]

(12)

\[
Mg \sum_{n=1}^{\infty} \int_0^L Y_p(x) \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(x) T_n(t) \, dx = Mg \sum_{n=1}^{\infty} \int_0^L Y_p(x) \delta(x - \beta) \, dx
\]

\[
= M \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) \int_0^L Y_p(x) \delta(x - \beta) \, dx
\]

\[
= M \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) Y_p(\beta)
\]

(13)

From orthogonality principle and from the orthogonal properties of the function \( Y_n(x) \), right side of equation (11) i.e.
\[
\sum_{n=1}^{\infty} \int_0^L Y_p(x) Y_n(x) S_n(t) \, dx = \sum_{n=1}^{\infty} S_n(t) \int_0^L Y_p(x) Y_n(x) \, dx
\]

(14)

Using (12), (13) and (14) the equation (11) reduces to
\[
Mg Y_p(\beta) - M \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) Y_p(\beta) = \sum_{n=1}^{\infty} S_n(t) Y_p(x)
\]

(15)

Equation (15) is re-written as
\[
S_p(t) = \frac{M}{Y_p} \left[ g - \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) \right] Y_p(\beta)
\]

(16)

From equation (5) and (6) we have
\[
E I \frac{\partial^4 y(x,t)}{\partial x^4} + pA \frac{\partial^2 y(x,t)}{\partial t^2} = \sum_{n=1}^{\infty} Y_n(x) S_n(t)
\]

(17)

Combining equation (16) and (17) we get,
\[
E I \frac{\partial^4 y(x,t)}{\partial x^4} + pA \frac{\partial^2 y(x,t)}{\partial t^2} = \sum_{n=1}^{\infty} Y_n(x) \frac{M}{V_n} \left[ g - \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_q(\beta) T_q(t) \right] Y_p(\beta)
\]

(18)

From equation (6) and (18) we have
\[
\frac{\partial^4}{\partial x^4} \left( \sum_{n=1}^{\infty} Y_n(x) T_n(t) \right) + pA \frac{\partial^2}{\partial t^2} \left( \sum_{n=1}^{\infty} Y_n(x) T_n(t) \right)
\]
Numerical Analysis

Numerical results for the displacement at the free end of a Mild Steel cantilever beam specimen (1500 x 50 x 6mm) for different positions of the moving mass such as; 1200 and 1848 gms respectively are obtained using the expression (6) with the help of a computer. These results are presented in Fig.2 to Fig.7. In Fig.2 a three dimensional graph has been plotted to present the variation of displacement at the free end of the beam with the velocity of the moving body vs. time. In Fig.3 and Fig.4 similar graphs have also been plotted for displacement at the free end for variation of mass of the moving body with time. In Fig.5 to Fig.7 displacement at the free end has been plotted against time for different velocities of the same moving mass.

By substituting the value of \( Y_n(x) \) from (7) into equation (20) we get

\[
- \sum_{n=1}^{\infty} Y_n(x) \frac{M}{V_n} \left[ g - \sum_{q=1}^{\infty} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 Y_q(\beta) T_q(t) \right] Y_n(\beta) = 0
\]

Equation (23) is a differential equation of II order, which can be solved by Runge-Kutta method. The contribution of \( (\frac{\partial}{\partial x})^2 \) is small when velocity 'V' is small.

\[
- \sum_{n=1}^{\infty} Y_n(x) \frac{M}{V_n} \left[ g - \sum_{q=1}^{\infty} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 Y_q(\beta) T_q(t) \right] Y_n(\beta) = 0
\]
Experimental Set-up

The experimental set-up used for performing the experiments is shown in schematic diagram (Fig. 12). A number of tests are conducted on a Mild steel beam specimen (1500 x 50 x 6 mm) with different moving masses such as 1200 and 1848 gms. as mentioned in numerical analysis, for determining dynamic deflection at the free end of the cantilever beam specimen. Experimental results of the free end deflection for various positions of the moving mass are recorded by positioning a LVDT pick-up at the free end. The LVDT pick-up is so arranged in the measuring system that it only records the instant displacement of the core, as the moving mass reaches a prefixed location of the cantilever beam specimen. These results for different velocities of the moving mass are plotted in Fig. 8 to Fig. 11. Corresponding numerical results are also presented in the same graph for comparison.

While conducting the experiments sufficient precautions have been taken in order to achieve the best possible accuracy in measurements.

Discussion

From the numerical analysis and experimental results, the following discussions are made:

1. In Fig. 2 comparisons are made among the displacements at the end point of the beam by changing the velocity of the moving mass from 31 KMPH to 120 KMPH. It is observed from Fig. 2, that as the velocity of the moving mass increases, the deflection at the end point of the beam decreases, which is a very peculiar phenomena. This is because of vibration of the beam in the higher modes which are exited due to the higher velocity of the moving mass.

2. From Figs. 3 to 7 displacements at the end point of the beam are found out by changing the mass of the moving body. From above figures it is found that as the mass of the moving body increases, the end deflection also increases. This is due to increase in the inertia of the moving mass.

3. From Figs. 8 to 11 comparisons are made between theoretical and experimental results for displacements at the end point of the beam. The authenticity of the theory developed is confirmed as the experimental results agree well with the theoretical results.
Conclusions

Based on the above discussions, the concluding remarks for the present investigation are depicted as follows:

1. Velocity of the moving mass has got a significant role in deciding the deflection of beam structures. With the increase in velocity, the dynamic deflection of beam structures show a decreasing trend.

2. At higher velocities of the moving mass, the higher modes of vibration are more active than the lower of vibration (1st mode). As a result the deflection of the beam decreases. However, with the increase in moving mass, the deflection also increases.

3. Further studies can be conducted by extending the present formulation for finding out dynamic deflection of complicated structures with moving masses used in actual practice.

References

VIBRATION CHARACTERISTICS OF A SPINNING SHAFT IN VISCOUS LIQUID

Dayal R. Parhi* and A.K. Behera**

Abstract

The analysis of vibration characteristics of a spinning cantilever shaft having a disk at the tip (rotating) in infinite fluid region is carried out in the present investigation. The virtual mass effect and the damping effect due to external fluid is incorporated in the ongoing analysis with the help of Navier Stokes Equation. It is observed that the damping effect is responsible mainly for reduction of vibration amplitude, while the virtual mass effect is responsible for both, 1) shifting the natural frequency of the rotor system and 2) for reducing the amplitude of vibration.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>Diameter of the shaft</td>
</tr>
<tr>
<td>δ</td>
<td>Whirling radius of the shaft</td>
</tr>
<tr>
<td>E</td>
<td>Modulus of elasticity of shaft material</td>
</tr>
<tr>
<td>e</td>
<td>Eccentricity</td>
</tr>
<tr>
<td>Fx, Fy</td>
<td>Fluid forces on rotor in x- and y- direction respectively</td>
</tr>
<tr>
<td>I</td>
<td>Section moment of inertia of the shaft</td>
</tr>
<tr>
<td>L</td>
<td>Total length of the shaft</td>
</tr>
<tr>
<td>m</td>
<td>Mass of the fluid displaced by the shaft per unit length (ρπR²)</td>
</tr>
<tr>
<td>ms</td>
<td>Mass of the shaft per unit length</td>
</tr>
<tr>
<td>m*</td>
<td>Dimensionless parameter (= m / ms)</td>
</tr>
<tr>
<td>M</td>
<td>Equivalent mass of fluid displaced by a rotor (= Ms1 + αeq s2)</td>
</tr>
<tr>
<td>M*</td>
<td>Dimensionless parameter (= M / M)</td>
</tr>
<tr>
<td>R1, R2</td>
<td>Radius of the shaft, Radius of the cylinder</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
</tr>
<tr>
<td>α, β</td>
<td>k R1, k R2</td>
</tr>
<tr>
<td>μ</td>
<td>Co-efficient of viscosity</td>
</tr>
<tr>
<td>ν</td>
<td>Co-efficient of kinematic viscosity</td>
</tr>
<tr>
<td>ρ</td>
<td>Fluid density</td>
</tr>
<tr>
<td>ωe</td>
<td>Natural angular frequency of the uncracked rotor in air</td>
</tr>
<tr>
<td>ω1</td>
<td>Fundamental natural frequency of a shaft (i.e without disk)</td>
</tr>
</tbody>
</table>

Introduction

In recent years there has been an increase in the use of very-very high speed rotors in viscous fluid, which requires a thorough vibrational analysis.

The dynamic analysis of spinning beams is of great interest to many researchers because of its wide applications in engineering problems. A number of practical structures such as spinning bodies in space application, high speed centrifuges, and rotating shaft in rotor dynamics can be modeled by such a system.

A systematic and comprehensive analysis on rotor dynamics can be found in the paper by Dirrentberg [1]. Bauer [2] carried out free as well as force vibration analysis on rotating shafts for all types of boundary conditions. Hashis and Sankar [3] highlighted the vibration analysis of spinning Timoshenko beam system under stochastic loading conditions with the help of finite element method. They have taken flexible rotor bearing system and solved it, taking into account the linear and nonlinear stiffnesses, and flexible bearing support. Lee et al. [4] proposed a modal analysis for a spinning Rayleigh beam. They presented the Galerkin's approach to analyze the forced response of an undamped gyroscopic effect. Katz et al. [5] used a finite integral transform technique to calculate transient response for spinning Rayleigh and Timoshenko beams with simply supported boundary conditions and subjected to a moving load.

Han and Zu [6] used a modal analysis technique which yields the transient response, eigen frequencies and mode shapes. In a subsequent paper Zu and Han [7] obtained a...
closed form solution for the free vibration analysis of spinning beams for all types of classical boundary conditions.

Keeping in view the above statements, the current investigation focuses on dynamic analysis of a cantilever rotor having disk at the tip, rotating in an infinite medium. Vibration motion of an externally damped system is significantly affected by the viscosity of the fluid. The resonant frequencies differ from the ones that occur in vacuum. This along with damping and virtual mass effects are analysed quantitatively in the present work. Comparison of numerical results with the experimental values available in literature [9] shows good agreement.

Equation for Fluid Velocities

A shaft of cross sectional radius $R_1$ rotating with a speed $w$ having whirling speed $\omega$ with $a$ as the radius, is shown in Fig. 1 (Fig. 1a and 1b).

The Navier-Stokes equation in polar co-ordinate is generally expressed as

$$
\frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left( \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial v}{\partial \theta} \right) - \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{\partial^2 u}{\partial r^2} - \frac{2 \partial u}{\partial r} \right)
$$

$$
\frac{\partial v}{\partial t} = \frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left( \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial u}{\partial \theta} \right) - \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{\partial^2 v}{\partial r^2} - \frac{2 \partial v}{\partial r} \right)
$$

(1)

In the above equation $u$ and $v$ denote the flow velocity in radial and tangential direction respectively and $p$ is the fluid pressure. With the help of stream function $\psi (r, \theta, t)$ the above equations can be written as;

$$
\nabla^2 \psi - \frac{1}{\psi} \frac{\partial}{\partial t} \nabla^2 \psi = 0
$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$

Equation (2) can be divided into two parts i.e.

$$
\nabla^2 \psi = 0, \quad \nabla^2 \psi - \left( \frac{1}{\psi} \right) \frac{\partial \psi}{\partial t} = 0
$$

(3)

The solution of equation (2) is expressed as;

$$
\psi = \psi_1 + \psi_2
$$

where $\psi_1$ and $\psi_2$ are solutions of equation (3).

The radial and tangential components of flow velocity at point $A$ in Fig. 1b are

$$
u_\rho = R_1 \omega \sin (\theta) - \delta \Omega \sin (\Omega t - \theta)$$

$$\nu_\theta = R_1 \omega \cos (\theta) + \delta \Omega \cos (\Omega t - \theta)
$$

(4)

where $\delta$ is the angle between $O'A$ and $OA$

$$\sin(\theta) = \frac{\delta}{R_1} \sin (\theta - \theta_0)$$

and $\cos (\theta) = 1$ for $\delta << R_1$

For $\tau = R_1$ equation (4) can be rewritten as ;

$$u \big|_{\tau = R_1} \delta (\omega - \Omega) \sin (\Omega t - \theta) = Re \left( -i(\omega - \Omega) e^{(i(\Omega t - \theta)} \right)$$

$$v \big|_{\tau = R_1} \delta (\omega - \Omega) \cos (\Omega t - \theta) = Re \left( \delta \Omega e^{(i(\Omega t - \theta)} + R_1 \omega \right)
$$

(5)

where $i = \sqrt{-1}$ and Re $[.]$ denotes the real part of $[.]$.

Taking $\omega = \Omega$ equation (5) reduces to

$$u \big|_{\tau = R_1} = 0, \quad v \big|_{\tau = R_1} = Re \left[ \delta \omega e^{(i(\omega - \theta)} \right] + R_1 \omega
$$

(6)

When the shaft is immersed in infinite extending fluid region, the boundary conditions $(r = R_2 = \infty)$ i.e the container radius $R_2$ is taken as $= \infty$ are taken as

$$u \big|_{r = R_2} = 0, \quad v \big|_{r = R_2} = 0
$$

(7)

Under these conditions, non-stationary components of the solutions $\psi_1$ and $\psi_2$ can be expressed as

$$\psi_1 (r, \theta, t) = F_1 (r) e^{(i(\omega t - \theta)}$$

$$\psi_2 (r, \theta, t) = F_2 (r) e^{(i(\omega t - \theta)}
$$

(8)

From the equation (8) and equation (3), we obtain

$$\frac{d^2 F_1}{dr^2} + \frac{1}{r} \frac{dF_1}{dr} - \frac{1}{r^2} F_1 = 0
$$

(9a)

$$\frac{d^2 F_2}{dr^2} + \frac{1}{r} \frac{dF_2}{dr} - \frac{1}{r^2} \left( \frac{1}{r^2} + k^2 \right) F_2 = 0
$$

(9b)

where $k = \sqrt{i \omega / \nu}$

Equation (9a) is Euler equation and equation (9b) is Bessel’s equation. The solution to the above equations can be written as

$$F_1 (r) = \delta \omega \left( A R_1^2 / r + B r \right)$$

$$F_2 (r) = \delta \omega R_1 \left( C l_1 (kr) + D K_1 (kr) \right)
$$

(10)
where A, B, C and D are arbitrary constants and \( I_1(kr) \) and \( K_1(kr) \) are modified Bessel's functions of 1st and 2nd kind respectively.

The non-stationary components of flow velocities can be written as

\[
\begin{align*}
\dot{u}_1 &= -\frac{1}{r} \frac{\partial y}{\partial \theta} e^{i\theta} A \left( \frac{R_i}{r} \right)^2 + B + C \frac{R_i}{r} I_1(kr) + \\
& \quad + D \frac{R_i}{r} K_1(kr) e^{i(\omega t - \theta)}
\end{align*}
\]

\[
\begin{align*}
\dot{v}_1 &= -\frac{1}{r} \frac{\partial y}{\partial \theta} e^{i\theta} \left[ -A \frac{1}{r^2} + B + C \left\{ \frac{R_i}{r} I_1(kr) + kR_i I_0(kr) \right\} + \\
& \quad + D \left\{ \frac{R_i}{r} K_1(kr) - kR_i K_0(kr) \right\} \right] e^{i(\omega t - \theta)}
\end{align*}
\]

Derivation of Fluid Forces Equation

Substituting the flow velocities given by equation (11) into equation (1), the non-stationary component of pressure \( p \) can be written as

\[
p = \oint \frac{\partial p}{\partial \theta} d\theta = \delta \rho \omega^2 \left( -A \frac{1}{r} R_i^2 + B r \right) e^{i(\omega t - \theta)}
\]

Normal stress \( \tau_r \) and tangential stress \( \tau_\theta \) due to flow can be written as

\[
\begin{align*}
\tau_r &= -p + 2\mu \frac{\partial v}{\partial r} \\
\tau_\theta &= \mu \left[ r \frac{\partial (\dot{v})}{\partial r} + \frac{1}{r} \frac{\partial \dot{u}}{\partial \theta} \right]
\end{align*}
\]

Fluid forces acting on the surface (i.e. \( r=R_i \)) per unit length of the shaft in the x and y direction are written as

\[
F_x = \int_0^{2\pi} \left( \tau_r \cos \theta - \tau_\theta \sin \theta \right) R_i d\theta
\]

\[
= m\delta \omega^2 \left( A - B - C_1(\alpha) - D K_1(\alpha) \right) e^{i\omega t}
\]

\[
F_y = \int_0^{2\pi} \left( \tau_r \sin \theta + \tau_\theta \cos \theta \right) R_i d\theta
\]

\[
= -im\delta \omega^2 \left( A - B - C_1(\alpha) - D K_1(\alpha) \right) e^{i\omega t}
\]

where \( \alpha = kR_i, \quad m = \rho \pi R_i^2 \)

Only the real parts of equation (14) has the meaning. So \( F_x \) and \( F_y \) after simplification can be expressed as

\[
\begin{align*}
F_x &= m\delta \omega^2 \left[ \text{Re}(H) \cos(\omega t) - \text{Im}(H) \sin(\omega t) \right] \\
F_y &= m\delta \omega^2 \left[ \text{Re}(H) \sin(\omega t) + \text{Im}(H) \cos(\omega t) \right]
\end{align*}
\]

where \( H=A-B-C_1(\alpha)-DK_1(\alpha) \) and \( \text{Re}(H), \text{Im}(H) \) denote the real and imaginary part of \( H \). The coordinates of the center of the shaft (as shown in Fig.1b) is \( x = \delta \cos \omega t, \ y = \delta \sin \omega t \). Now

\[
\begin{align*}
F_x &= -m \text{Re}(H) \frac{d^2 x}{dt^2} + m\omega \text{Im}(H) \frac{dx}{dt} \\
F_y &= -m \text{Re}(H) \frac{d^2 y}{dt^2} + m\omega \text{Im}(H) \frac{dy}{dt}
\end{align*}
\]

In equation (16), \( m \text{Re}(H) \) denotes the virtual or added mass of the fluid relating to the inertia force of the shaft while \( -m\omega \text{Im}(H) \) denotes the viscous damping co-efficient.

Expression of Amplitude for the Governing System

In the current analysis a lumped mass at the tip of the cantilever type rotating shaft immersed in infinite fluid region is considered.

An equivalent lumped mass of a rotating shaft is given by \( K_s/\omega_0^2 \), where \( K_s \) and \( \omega_0 \) are the stiffness and fundamental natural frequency of the shaft respectively.

The ratio of the equivalent lumped mass to the total mass of the shaft is given by the expression;

\[
\alpha_{eq} = \frac{K_s}{\omega_0^2 M_{eq}} \quad \text{where } M_{eq} \text{ is the mass of the shaft.}
\]

If a disk with mass \( M_{d1} \) is attached at the tip of the shaft, a total lumped mass of the rotor becomes;

\[
M_s = M_{d1} + \alpha_{eq} M_{d2}
\]

The equation of motion of an equivalent single degree of freedom system of the rotor in fluid is reduced to

\[
\begin{align*}
M_s \frac{d^2 x}{dt^2} (x + \epsilon \cos \omega t) + K_s x &= F_x \\
M_s \frac{d^2 y}{dt^2} (y + \epsilon \sin \omega t) + K_s y &= F_y
\end{align*}
\]

The fluid forces from equation (16) can be written as
\begin{align*}
F_x &= -M \Re(H) \frac{d^2 x}{dt^2} + M \omega \Im(H) \frac{dx}{dt} \\
F_y &= -M \Re(H) \frac{d^2 y}{dt^2} + M \omega \Im(H) \frac{dy}{dt} \tag{18}
\end{align*}

where
\begin{align*}
M \Re(H) &= M_1 \Re(H_1) + \alpha_{eq} M_2 \Re(H_2) \\
M \Im(H) &= M_1 \Im(H_1) + \alpha_{eq} M_2 \Im(H_2) \\
M &= M_1 + \alpha_{eq} M_2
\end{align*}

where \( M_1 \) and \( M_2 \) mass of the fluid displaced by the disk and shaft respectively.

From equation (17) and (18) we have
\begin{align*}
((M_s + M \Re(H))^2 - M \omega \Im(H) \frac{dx}{dt} + K_s x \\
= M_s \omega^2 \cos \omega t \\
((M_s + M \Re(H))^2 - M \omega \Im(H) \frac{dy}{dt} + K_s y \\
= M_s \omega^2 \sin \omega t \tag{19}
\end{align*}

Equation (19) in dimensionless form can be written as
\begin{align*}
1 + M^* \Re(H) \frac{d^2 \xi}{dt^2} - M^* \omega^* \Im(H) \frac{d\xi}{dt} + \xi = \xi^* (\omega^*)^2 \cos (\omega^* t) \\
1 + M^* \Re(H) \frac{d^2 \eta}{dt^2} - M^* \omega^* \Im(H) \frac{d\eta}{dt} + \eta = \eta^* (\omega^*)^2 \sin (\omega^* t) \tag{20}
\end{align*}

where \( \xi = x/R_1, \eta = y/R_1, \omega^* = \omega/\omega_1, \tau^* = t/R_1 \)
\( M^* = M/M_1, \omega_1 = \omega_1/\omega_1, \tau = \sqrt{K_s/M_1} \)

The steady state solution of the above equation can be obtained in dimensionless form as
\begin{align*}
\xi^* = \delta^* \cos (\omega^* \tau - \phi), \text{ where } \delta^* = \delta^0/R_1 \tag{21}
\end{align*}

and
\begin{align*}
\delta^0 = \frac{A}{\sqrt{|k - (\omega^*)^2|^2 + |C \omega^*|^2}}, \Phi = \tan^{-1} \left[ \frac{C \omega^*}{K - (\omega^*)^2} \right] \\
C = -\frac{M^* \omega^* \Im(H)}{1 + M^* \Re(H)}, K = \frac{1}{1 + M^* \Re(H)} \quad A = \frac{\varepsilon^* (\omega^*)^2}{1 + M^* \Re(H)} \tag{22}
\end{align*}

where \( \delta^0 \) is the minimum dimensionless amplitude.

**Numerical Analysis and Discussions**

In the current investigation, the rotor system being used has got the following specifications.

1. Mild Steel cantilever shaft rotating in viscous fluid (infinite region) with a disk at the end span.
2. Length of the cantilever shaft, \( 'L' = \text{varies (1.3, 1.6 and 1.8m.)} \)
3. Radius of the cantilever shaft, \( 'R_1 = 0.012\text{m}. \)
4. Radius of the disk, \( 'R_{disk} = 0.04\text{m}. \)
5. Length of the disk, \( 'L_{disk} = 0.04\text{m}. \)
6. Mass of the disk, \( 'M_{disk} = 0.04\text{m}. \)
7. Damping Co-efficient, \( '\nu = \text{varies (2.3, 0.427 and 0.0633 Stokes)} \)
8. Equivalent mass of fluid displaced/Corresponding mass of the shaft \( 'm^* = \text{varies (0.158, 0.1534 and 0.144)} \)

The configuration of the system used for the ongoing analysis is given in Fig.1a and Fig.1b. Fig.1a shows the
Fig. 2 Frequency Ratio Vs Dimensionless Amplitude Ratio
Mild Steel Shaft (Ri=0.012m., L=1.3m.)
cantilever rotor with disk rotating in an infinite fluid region.

Fig. 2 and 3 shows the dimensionless amplitude of vibration with respect to frequency ration in various viscous media. It is observed that as the viscosity of the fluid increases the amplitude of vibration and resonant frequency decreases.

Another effect is seen, when the two figures 2 and 3 compared for a particular viscous fluid is that, the dimensionless amplitude is smaller in case of rotor with a larger shaft. The above phenomenon is because of higher value of virtual mass coefficient (|mRe(H)|) and damping coefficient (| - mIm(H) |) in rotor system with the longer shaft.

The increase in virtual mass effect (|mRe(H)|) at constant damping effect (| - mIm(H)|) decreases the critical speed and also decreases of amplitude of vibration. This effect can be visualized from Fig.4.
From Fig. 5 it is observed that the damping effect \((I - \omega_0 \text{Im}(H))\) at constant virtual mass effect \((|\text{Im}(\text{Re}(H))|)\) is responsible for reducing the amplitude of vibration, without shifting the critical speed to a noticeable extent.

In Fig. 6 a comparison made between the numerical results and experimental results obtained from Walston [9].

For experimental verification the dimensions taken for cantilever rotor are:

- Length of shaft = 0.762 m.
- Radius of shaft = 0.00635 m.
- Weight of the disk = 1.3 Kg.
- Radius of the disk = 0.054 m.
- Liquid medium 70% and 100% Glycerin.

Conciliation is seen between the theoretical and experimental results.

**Conclusions**

The viscosity effect due to external fluid has a significant influence, i.e., as the viscosity of the fluid increases, the amplitude of vibration decreases. This is because of increase in damping effect \((I - \omega_0 \text{Im}(H))\) and virtual mass effect \((|\text{Im}(\text{Re}(H))|)\).

The virtual mass effect \((|\text{Im}(\text{Re}(H))|)\) at constant damping effect \((I - \omega_0 \text{Im}(H))\) reduces the amplitude of vibration and also decreases the resonant frequency. But damping effect \((I - \omega_0 \text{Im}(H))\) at constant virtual mass effect \((|\text{Im}(\text{Re}(H))|)\) is mainly responsible for decreasing the amplitude of vibration.

The investigation done here can be utilized for condition monitoring of rotor systems in viscous fluid and in finding out the amplitude and critical speed of vibration of high speed rotors rotating in viscous fluid.

**References**

DYNAMIC CHARACTERISTICS OF CANTILEVER BEAM WITH TRANSVERSE CRACK

Dayal R. Parhi*, A.K. Behera* and R.K. Behera*

Abstract

It has been established since long, that the dynamic response of a structure changes due to presence of crack. Scientific analysis of such phenomena can be utilized for damage diagnosis and detection of crack in structures. The present investigation is an attempt in that direction. Theoretical expressions have been developed for determining natural frequencies and mode shapes for elastic cantilever beam vibration with a transverse crack using flexibility influence coefficients and local stiffness matrix. The numerical results are compared with the experimental ones in order to establish the theory. It has also been observed from numerical results that there is appreciable difference between dynamic response of cantilever beam with and without crack, which can be utilized for crack identification.

Nomenclature

\( a_i \) = depth of crack  
\( A \) = cross-sectional area of the beam  
\( A_{ij}, i=1,2 \) = unknown co-efficients of matrix A  
\( B \) = width of the beam  
\( \mathbf{B} \) = vector of exciting motion  
\( C_0 \) = \( \frac{E}{\rho} \frac{y_i}{y} \)  
\( C_y \) = \( \frac{E L}{\mu} \frac{y_i}{y} \)  
\( E \) = young’s modulus of elasticity of the beam material  
\( F_{ij}, i=1,2 \) = experimentally determined function  
\( i, j \) = \( i, j \) = variables  
\( J \) = strain-energy release rate  
\( K_{ij}, i=1,2 \) = stress intensity factors for \( P_i \) loads  
\( K_x \) = \( \frac{\omega L}{C_x} \)  
\( K_y \) = \( \frac{\omega L}{C_y} \)  
\( K_{ij} \) = local flexibility matrix elements  
\( L \) = length of the beam  
\( L_i \) = location of the crack from fixed end  
\( M_{ij} \) = compliance constant  
\( M_i \) = \( \frac{M_i}{M} \)  
\( P_i, i=1,2 \) = axial force \((i=1)\), bending moment \((i=2)\)  
\( Q \) = stiffness matrix for free vibration  

* = = = = = = = = =

\( \omega_i, i=1,2 \) = normal functions (longitudinal) \( u_i(x) \)  
\( x \) = co-ordinate of the beam  
\( y \) = co-ordinate of the beam  
\( Y_a \) = amplitude of the exciting vibration  
\( y_j, i=1,2 \) = normal functions (transverse) \( y_j(x) \)  
\( W \) = depth of the beam  
\( \omega \) = natural circular frequency  
\( \beta \) = relative crack location \( \frac{L_i}{L} \)  
\( \mu \) = \( C \rho \)  
\( \rho \) = mass-density of the beam  
\( \xi_i \) = relative crack depth \( \frac{a_i}{w} \)  

Introduction

The characteristics of transverse vibration of elastic structures with crack has been studied for decades. Natural frequencies and mode shapes undergo variation due to presence of crack, variation in its local and intensity. A number of investigators are working round the globe on various aspects of cracked structures.

Adams et al [1] studied both analytically and experimentally the natural frequencies of longitudinal vibration of a free-free bar with a localised crack, modeled by a linear spring of infinitesimal length separating the two sections of the bar.

* Department of Mechanical Engineering, Regional Engineering College, Rourkela, Orissa - 769 008, India
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F.D. Ju et al [2] made an extensive study on diagnosis of fracture damage in structures. They developed the concept of 'fracture hinge' analytically and applied the same to a cracked section for detecting fracture damage in simple structures. They have verified experimentally that the structural effect of a cracked section can be represented by an equivalent spring loaded hinge and the principle of fracture diagnosis in structures.

Mayes and Davies [3] used finite element method to formulate the vibration response of a rotor with a transverse crack. Experiments were also conducted by them for comparison. Gudmundson [4] developed an expression using perturbation technique to find the variation in natural frequencies for free-free bar with a transverse crack at the center. He has reported that his results are comparable with the values obtained by finite element analysis.


Dentsoras and Dimarogonas [6] studied the vibration response of a cantilever bar with a crack at the fixed end, subjected to longitudinal harmonic force at free end. Papadopoulos and Dimarogonas [7] made extensive studies on shafts for cracks identification. Reviewing a number of publications free vibrational behaviour of such shafts are analyzed, by coupling longitudinal and bending modes. The applicability of the method for crack identification has been demonstrated.

In order to utilize the findings in actual practice, a systematic approach has been adopted in the present investigation to develop theoretical expressions for evaluation of natural frequencies and mode shapes of simple elastic cantilever beams with transverse crack. Experiments has been conducted to validate the theory developed. Natural frequencies and the mode shapes of the cracked specimen are found out both numerically and experimentally for different relative crack depth and relative crack position from free to fixed end of the cantilever beam. It is observed that there is appreciable variation in dynamic characteristic of the beam with different crack configuration and there is an abrupt change in mode shape at the crack position. Such variation in dynamic characteristics in cantilever beams with a crack as compared to that of similar uncracked beam can be utilized for detection of crack.

Local Flexibility of a Cracked Beam under Bending and Axial Loading

The presence of a transverse surface crack of depth \(a_1\) on a beam of width 'B' and height 'W' introduces a local flexibility, which can be defined in matrix form, the dimension of which depends on the degree of freedom.

![Fig. 1: Geometry of Beam, (a) Cantilever Beam, (b) Cross-Sectional View of the Beam, (c) Segments taken during Integration at the Crack Section](image-url)
Here a 2x2 matrix is considered. A cantilever beam is subjected to axial force ($P_1$) and bending moment ($P_2$) (as shown in Fig 1a) which gives coupling with the longitudinal and transverse motion.

The strain energy release rate at the fractured section can be written as [8]:

$$J = \frac{1}{E'} (K_{11} + K_{12})^2$$

where

$$\frac{1}{E'} = 1 - \frac{\nu^2}{E} \quad \text{(for plane strain condition)}$$

$$= \frac{1}{E} \quad \text{(for plane stress condition)}$$

The values of stress intensity factors from earlier studies [8] are

$$K_{11} = \frac{P_1}{BW \sqrt{\pi a}} \left( F_1 \left( \frac{a}{W} \right) \right), \quad K_{12} = \frac{6P_2}{BW^2 \sqrt{\pi a}} \left( F_2 \left( \frac{a}{W} \right) \right)$$

where expressions for $F_1$ and $F_2$ are as follows

$$F_1 \left( \frac{a}{W} \right) = \left( \frac{2W}{\pi a} \tan \left( \frac{\pi a}{2W} \right) \right)^{0.5}$$

$$\left\{ 0.752 + 2.02 \left( \frac{a}{W} \right) + 0.37 \left( 1 - \sin \left( \frac{\pi a}{2W} \right) \right)^3 \right\}$$

$$\cos \left( \frac{\pi a}{2W} \right)$$

$$F_2 \left( \frac{a}{W} \right) = \left( \frac{2W}{\pi a} \tan \left( \frac{\pi a}{2W} \right) \right)^{0.5}$$

$$\left\{ 0.923 + 0.199 \left( 1 - \sin \left( \frac{\pi a}{2W} \right) \right)^4 \right\}$$

$$\cos \left( \frac{\pi a}{2W} \right)$$

Let $U_1$ be the strain energy due to the crack. Then from Castigliano's theorem, the additional displacement along the force $P_1$ is:

$$u_i = \frac{\partial U_1}{\partial P_1} \quad \text{(1)}$$

The strain energy will have the form

$$U_i = \int \frac{\partial U_i}{\partial a} da = \int J da$$

where $J = \frac{\partial U_i}{\partial a}$ is the strain energy density function

From (1) and (2), thus we have

$$u_i = \frac{\partial U_i}{\partial P_1} \left[ J \left( a \right) da \right] \quad \text{(3)}$$

The flexibility influence co-efficient $C_{ij}$ will be, by definition

$$C_{ij} = \frac{\partial u_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j} \int J \left( a \right) da$$

To find out the final flexibility matrix we have to integrate over the breadth 'B'

$$C_{ij} = \int \int J \left( a \right) da dz$$

Putting the value strain energy release rate from above eqn (5) modifies as

$$C_{ij} = \frac{B}{E'} \frac{\partial^2}{\partial P_i \partial P_j} \int \left( K_{11} + K_{12} \right)^2 da$$

Putting $\xi = (a/W), \xi = \frac{da}{W}$ and when $a=0$, $\xi = 0$; $a=a_1, \xi = a_1/W = \xi_1$

From the above condition, eqn. (6) converts to

$$C_{ij} = \frac{B}{E'} \frac{\partial^2}{\partial P_i \partial P_j} \frac{\xi}{\xi_1} \int \left( K_{11} + K_{12} \right)^2 \xi d\xi$$

From the equation (7), calculating $C_{11}, C_{12} (= C_{21})$ and $C_{22}$ we get
Analysis of Vibration Characteristics of the Cracked Beam

Free Vibration

A cantilever beam of length 'L' width 'B' and depth 'W', with a crack of depth 'a1' at a distance 'L1' from the fixed end is considered (as shown in Fig. 1). Taking \( U_1(x,t) \) and \( U_2(x,t) \) as the amplitudes of longitudinal vibration for the sections before and after the crack and \( Y_1(x,t), Y_2(x,t) \) are the amplitudes of bending vibration for the same sections (shown in Fig. 3).

the local stiffness matrix can be obtained by taking the inversion of compliance matrix, i.e.

\[
K = \begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}^{-1}
\]

Fig (2) shows the variation of dimensionless compliance to that of crack depth.

\[
\begin{align*}
C_{11} &= \frac{B W}{E I} \int_0^{\xi_1} \frac{na}{B^2 W^2} 2 (F_1(\xi))^2 \, d\xi \\
&= \frac{2\pi}{B E I} \int_0^{\xi_1} (F_1(\xi))^2 \, d\xi \\
C_{12} &= C_{21} = \frac{12\pi}{E' B W} \int_0^{\xi_1} F_1(\xi) F_2(\xi) \, d\xi \\
C_{22} &= \frac{12\pi}{E' B W^2} \int_0^{\xi_1} F_2(\xi) F_2(\xi) \, d\xi
\end{align*}
\]

Converting the influence co-efficient into dimensionless form

\[
\begin{align*}
C_{11}' &= \frac{C_{11}}{2\pi} \\
C_{12}' &= C_{12} \frac{E' B W}{12\pi} = C_{21} \\
C_{22}' &= C_{22} \frac{E' B W^2}{72\pi}
\end{align*}
\]

The normal function for the system can be defined as

\[
U_2(x) = A_1 \cos(K_{uy} x) + A_2 \sin(K_{uy} x) \\
Y_1(x) = A_3 \cos(K_{y1} x) + A_4 \sin(K_{y1} x) + A_5 \cos(K_{y2} x) + A_6 \sin(K_{y2} x)
\]

\[
Y_2(x) = A_7 \cos(K_{y2} x) + A_8 \sin(K_{y2} x) + A_9 \cos(K_{y3} x) + A_{10} \sin(K_{y3} x) + A_{11} \cos(K_{y4} x) + A_{12} \sin(K_{y4} x)
\]

where \( x = \frac{X}{L'}, u = \frac{U}{L'}, y = \frac{Y}{L'}, \beta = \frac{L_1}{L} \)

\[
K_{uy} = \frac{a L}{C_u}, C_u = (\frac{E}{\rho})^{1/2}, K_{y1} = (\frac{a L}{C_y})^{1/2}, C_y = (\frac{E}{\rho})^{1/2}, \mu = A \rho
\]

\( A_i, (i = 1, 12) \) Constants are to be determined, from boundary conditions.

The boundary conditions of the cantilever beam in consideration are:

\[
\frac{dU_1}{dx} = 0, \quad U_1(x=0) = 0
\]

\[
\frac{dY_1}{dx} = 0, \quad Y_1(x=0) = 0
\]

\[
\frac{dY_2}{dx} = 0, \quad Y_2(x=0) = 0
\]

\[
U_2(x=L) = 0, \quad Y_2(x=L) = 0
\]

\[
Y_2(x=L) = 0, \quad Y_2(x=L) = 0
\]
At the Cracked Section:

\[ \begin{align*}
\tilde{u}_1(\beta) &= \tilde{u}_2(\beta) \\
\tilde{y}_1(\beta) &= \tilde{y}_2(\beta) \\
\tilde{y}_1'(\beta) &= \tilde{y}_2'(\beta) \\
\tilde{y}_1''(\beta) &= \tilde{y}_2''(\beta)
\end{align*} \]

(21)

(22)

(23)

(24)

Also at the cracked section, we have:

\[ AE \frac{d\tilde{u}_1(L_1)}{dx} = K_{11}(\tilde{u}_2(L_1) - \tilde{u}_1(L_1)) \]

\[ + K_{12} \left( \frac{d\tilde{y}_2(L_1)}{dx} - \frac{d\tilde{y}_1(L_1)}{dx} \right) \]

Multiplying both sides by \( \frac{AE}{L K_{11} K_{12}} \) simplifying we get,

\[ M_1 M_2 \tilde{u}_1'(\beta) = M_2 (\tilde{u}_2(\beta) - \tilde{u}_1(\beta)) + M_1 \tilde{y}_2'(\beta) - \tilde{y}_1'(\beta) \]

(25)

Similarly,

\[ EI \frac{d^2\tilde{y}_1(L_1)}{dx^2} = K_{21}(\tilde{u}_2(L_1) - \tilde{u}_1(L_1)) \]

\[ + K_{22} \left( \frac{d\tilde{y}_2(L_1)}{dx} - \frac{d\tilde{y}_1(L_1)}{dx} \right) \]

Multiplying both sides by \( \frac{EI}{L^2 K_{22} K_{21}} \) & simplifying we get,

\[ M_3 M_4 \tilde{y}_1''(\beta) = M_3 (\tilde{u}_2(\beta) - \tilde{u}_1(\beta)) + M_4 \tilde{y}_2''(\beta) - \tilde{y}_1''(\beta) \]

(26)

Where

\[ M_1 = \frac{AE}{L K_{11}}, \quad M_2 = \frac{AE}{K_{12}}, \quad M_3 = \frac{EI}{L K_{21}}, \quad M_4 = \frac{EI}{L^2 K_{21}} \]

The normal functions, eqn(14) along with the boundary conditions as mentioned from expression (15) to (26), yield the characteristic equation of the system as:

\[ |Q| = 0 \]

(27)

This determinant is a function of natural circular frequency (\( \omega_n \)), the relative location of the crack (\( \beta \)) and the local stiffness matrix (\( K \)) which in turn is a function of the relative crack depth (\( a_1/W \)).

**Forced Vibration**

If the cantilever beam with a transverse crack is excited at its free end by a harmonic excitation \( \ddot{Y} = Y_0 \sin \omega t \), the non-dimensional amplitude at the free end may be expressed as \( \ddot{y}_2(1) = \frac{y_0}{L} = \ddot{y}_0 \). Therefore the boundary conditions for the beam remain same as before (i.e. the expressions from 15 to 26) except the expression (20) which modifies as \( \ddot{y}_2(1) = \ddot{y}_0 \).

The constants \( A_i, i = 1, \ldots, 12 \) are then computed from the algebraic condition \( Q_1 D = b \)

(28)

\[ Q_1 \]

is the \((12 \times 12)\) matrix obtained from boundary conditions as mentioned above,

\[ D \]

is a column matrix obtained from the constants,

\[ B \]

is a column matrix, transpose of which is given by,

\[ B^T = [0 0 0 \ddot{y}_0 0 0 0 0 0 0] \]

**Numerical Analysis**

The numerical results for relative amplitude of transverse vibration at different locations of Aluminium cracked specimens \((800 \times 50 \times 6 \text{ mm})\) are obtained using theoretical expressions \((14(c)/14(d))\) with the help of computer. These results are presented in Fig.4 to Fig.6. Similar results for uncracked specimen are also plotted for immediate comparison. The logarithm value of characteristic equation \(Q\) with respect to frequency ratio \( \omega / \omega_0 \) is plotted in Fig.7, in order to evaluate the natural frequencies of the specimen. Numerical results for dimension-less frequency \( \omega_0 / \omega_0 \) of the same Aluminium specimen due to different relative crack depth and relative crack positions are also presented in Fig.8 and Fig.9.
Experimental Set-up

An experimental set-up used for performing the experiments is shown in schematic diagram (Fig. 16). A number of tests are conducted on Aluminium beam specimen (800 x 50 x 6mm) with a transverse crack for determining the natural frequencies and mode shapes for different crack depths. Experimental results of amplitude of transverse vibration at various locations along the length of the beam are recorded by positioning the vibration pick-up and tuning the vibration generator at the corresponding resonant frequencies. These results for first three modes are plotted in Fig.10 to Fig.12. Corresponding numerical results are also presented in the same graph for comparison. Experimental values for acceleration are also recorded with the help of an accelerometer, setting the vibration generator at different frequencies. These results are plotted in Fig.13 to Fig. 15 for different relative crack depth.
Fig. 4c  Relative Amplitude vs. Relative Distance ($x$) from the Fixed end of the Cantilever Beam (3rd Mode of Vibration), $a_1 / W = 0.166$, $L_1 / L = 0.125$, . . . With Crack, ---- Without Crack

Fig. 5a  Relative Amplitude vs. Relative Distance ($x$) from the Fixed end of the Cantilever Beam (1st Mode of Vibration), $a_1 / W = 0.334$, $L_1 / L = 0.125$, . . . With Crack, ---- Without Crack

Fig. 4c1  Magnified View of Fig. 4c in the Range $0.12488 \leq \frac{x}{10} \leq 0.12513$

Fig. 5a1  Magnified View of Fig. 5a in the Range $0.12488 \leq \frac{x}{10} \leq 0.12513$
Relative amplitude \( \times 10 \)

**Fig. 5b** Relative Amplitude vs. Relative Distance \( \tilde{x} \) from the Fixed end of the Cantilever Beam (2nd Mode of Vibration), \( a_1 / W = 0.334 \), \( L_1 / L = 0.125 \ldots . \) With Crack, \(-\) Without Crack

**Fig. 5c** Relative Amplitude vs. Relative Distance \( \tilde{x} \) from the Fixed end of the Cantilever Beam (3rd Mode of Vibration), \( a_1 / W = 0.334 \), \( L_1 / L = 0.125 \ldots . \) With Crack, \(-\) Without Crack

**Fig. 5b1** Magnified View of Fig. 5b in the Range \( 0.12488 \leq \tilde{x} \leq 0.12513 \)

**Fig. 5c1** Magnified View of Fig. 5c in the Range \( 0.12488 \leq \tilde{x} \leq 0.12513 \)
Fig. 6a Relative Amplitude vs. Relative Distance $x$ from the Fixed end of the Cantilever Beam (1st Mode of Vibration), $a_1 / W = 0.50$ $L_1 / L = 0.125$, ...... With Crack, ------ Without Crack

Fig. 6b Relative Amplitude vs. Relative Distance $x$ from the Fixed end of the Cantilever Beam (2nd Mode of Vibration), $a_1 / W = 0.50$ $L_1 / L = 0.125$, ...... With Crack, ------ Without Crack

Fig. 6a1 Magnified View of Fig. 6a in the Range $0.12488 \leq x \leq 0.12513$

Fig. 6b1 Magnified View of Fig. 6b in the Range $0.12488 \leq x \leq 0.12513$
Fig. 6c: Relative Amplitude vs. Relative Distance (x) from the Fixed end of the Cantilever Beam (3rd Mode of Vibration), $a_1/W = 0.50$ 
$L_1/L = 0.125$, ... With Crack, --- Without Crack

Fig. 7: Natural Logarithm of Values of the Characteristic Determinant ($Q$) vs. Dimensionless Frequency ($\omega/\omega_0$), $a_1/W = 0.5$, $L_1/L = 0.125$, ... With Crack, --- Without Crack

Fig. 6c1: Magnified View of Fig. 6c in the Range $0.12408 < x < 0.12513$

Fig. 8: Dimensionless Frequency vs. Relative Crack Depth of the Cantilever Beam $L_1/L = 0.0625$ ($\omega_c$ and $\omega_0$ are the Eigen Frequencies for Crack and Uncrack Beam respectively)
Fig. 9 Dimensionless Frequency vs. Relative Crack Position of the Cantilever Beam $L_1/W = 0.06$ ($\omega_2$ and $\omega_0$ are the Eigen Frequencies for Crack and Uncrack Beam respectively).

Fig. 10 Relative Amplitude vs. Relative Distance ($x$) from the Free end of the Cantilever Beam (1st Mode of Vibration) $a_1/W = 0.5$, $L_1/L = 0.125$. ---- Experimental Values  ____ Theoretical (Numerical Values)

Fig. 11 Relative Amplitude vs. Relative Distance ($x$) from the Free end of the Cantilever Beam (2nd Mode of Vibration) $a_1/W = 0.5$, $L_1/L = 0.125$. ---- Experimental Values  ____ Theoretical (Numerical Values)

Fig. 12 Relative Amplitude vs. Relative Distance ($x$) from the Free end of the Cantilever Beam (3rd Mode of Vibration) $a_1/W = 0.5$, $L_1/L = 0.125$. ---- Experimental Values  ____ Theoretical (Numerical Values)
Fig. 13 Acceleration vs. excited Frequency Experimental Results for Aluminium Beam

Fig. 14 Acceleration vs. excited Frequency Experimental Results for Aluminium Beam

Fig. 15 Acceleration vs. excited Frequency Experimental Results for Aluminium Beam
Fig. 16 Schematic Block Diagram of Experimental Set-up

Discussion

From the numerical analysis and experimental results the following points may be discussed:

It is evident from Fig. 4(a) to Fig. 4(c) that up to a relative crack depth of 0.166 and relative crack position 0.125 there is no appreciable change in mode shapes as compared with similar uncracked beam. However, with the magnification of ordinates at the vicinity of the crack position as shown in Fig. 4(a1) to Fig. 4(c1) significant variations is noted. Similarly for relative crack depth of 0.334 these variation are more prominent in all the 1st three modes as evident in Fig. 5(a) to Fig. 5(c1). Again with the increase in the relative crack depth (0.50), keeping the relative crack position same as before there is an appreciable variation in the 1st and 2nd mode shapes, as depicted in Fig. 6(a) to Fig. 6(b1), where as in the third mode this difference is quite small as shown in Fig. 6(c). However, with magnification of ordinates as shown in Fig. 6(c1), abrupt changes in mode shape is visible at the location of the crack.

From Fig. 7, it is evident that there is an appreciable variation between the natural frequency of cracked and uncracked cantilever beams. With the increase in mode of vibration this difference increases. It is observed from Fig. 8 that the transverse mode fundamental frequency decreases with the increase in relative crack depth. However, the same fundamental frequency increases with the increase in relative crack position as evident from Fig. 9.

From the Fig. 10 to Fig. 12 it is observed that the experimental values of natural frequencies decrease with the increase in relative crack depth, for all modes of vibration which is also evident in theoretical analysis (Fig. 6).

Conclusions

Relative crack depth as well as relative crack position on a structure have got significant influence on its dynamic characteristics. The relative crack depth up to \( \alpha_1/w = 0.166 \), with relative crack position \( L_1/L \approx 0.125 \) does not show any remarkable change in the mode shapes of the cantilever beam with a transverse crack in comparison with that of uncracked ones. But for higher relative crack depth, there will be significant variation in mode shapes.

The natural frequency of a cantilever beam with a crack decreases with the increase in relative crack depth as well as with the shift of relative crack position from free end to fixed end of the beam.

Further study can be extended to utilize these informations for analysing vibrational behaviour of cracked structure in viscous medium.
References


April 25, 1999

Dr. D. Parhi
Flat-3*
2 Claude Place
OFF Albany Road
CARDIFF - CF2 3QF
WALES, UK

Re:98-CSME-17: “Vibration analysis of cantilever type cracked rotor in viscous fluid”
by Parhi & Behera

Dr. Parhi,

I am pleased to inform you that your paper is acceptable for publication in its present form. Your paper is scheduled to appear in Vol. 23, No. 3, due out July 1999.

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VIBRATION ANALYSIS OF CANTILEVER TYPE CRACKED ROTOR IN VISCOUS FLUID

ABSTRACT

The increasing utilisation of rotors in industries and other fields require interpretation of their vibration characteristics and failure conditions. Therefore, computer programmes that have been used will have to be constantly updated. This paper deals with the cantilever type rotor containing a transverse crack subjected to viscous medium. Damping effect and virtual mass effect are taken into account through Navier Stokes equation. The stiffness of the cracked shaft is evaluated from the theory of fracture mechanics. For analysis rotor with different crack depth, subjected different viscous medium is considered. The numerical result obtained from theoretical analysis is compared with the experimental results, which agree well.