CHAPTER-4
For several decades, engineers are investigating the potential hazard produced due to moving masses on structures. The dynamic response of structures carrying moving masses is a problem of widespread practical significance. It has also been observed that the structures and machine parts generally fail prematurely due to the presence of crack. The frequency of such failure multiplies several times, in case of structures with cracks but subjected to moving mass.

From the literature survey presented section 2.5 of chapter-2, it has been observed that no significant specific investigation has been carried out for dynamic characteristics of cracked beam subjected to moving mass. Although the theory for dynamic response of uncracked beam carrying moving mass has been developed but no experimental verifications for the theory have been reported. Keeping all these in view, theoretical analysis have been developed to determine dynamic response of cracked structures subjected to moving mass.

The theoretical analysis has been carried out for three cases, starting with dynamic response of a solid cantilever beam with moving mass. Other two cases are; dynamic response of cracked cantilever beam with a moving mass and dynamic response of simply supported cracked shaft subjected to moving mass.
Fig. 4.2.1. Beam with moving mass.
z is the point of interest where the displacement of the beam is to be considered.
The effect of magnitude and velocity of the moving mass and the position of the crack on dynamic behavior of the corresponding system has been thoroughly investigated. Using continuum mechanics, differential equations have been developed for the system, which are subsequently solved by Runge-Kutta technique and are used to find out the dynamic deflection of the system. Dynamic deflection between cracked and uncracked shaft are compared and the results show a significant variation. Theoretical results are also compared with the experimental results, which show very-good agreement.

4.2 DYNAMIC RESPONSE OF BEAM WITH MOVING MASS

Dynamic response of cantilever beam with a moving mass has been developed using the theoretical analysis as presented below in detail. Numerical results have also been generated from the theoretical expressions developed in the analysis, for comparison.

4.2.1. THEORETICAL ANALYSIS

The equation of motion of an uniform beam having a mass 'm' per unit length subjected to a moving mass 'M', can be written as [125];

\[
EI \frac{\partial^4 y(x,t)}{\partial x^4} + \mu \frac{\partial^2 y(x,t)}{\partial t^2} = P(x,t)
\]  

(4.2.1)

Neglecting damping in the system, where P(x,t) is the External force which can be taken as [104];
\[ P(x,t) = F(t) \delta(x-vt) \quad (4.2.2) \]

and

\[ F(t) = Mg - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 y(\beta,t) \quad \text{where}, \beta = vt \quad (4.2.3) \]

Where \( F(t) \) is the force due to moving mass at that instant and \( \delta \) is the Dirac-delta function.

The Dirac-delta function \( \delta(x - x_0) \) has the following properties:

- \( \delta(x - x_0) = 0 \) for all \( x \neq x_0 \)
- = greater than any assumed value for \( x = x_0 \)

\[ \int_0^\infty f(x) \delta(x - x_0) \, dx = f(x_0) \quad 0 < x_0 < \infty \]

Then for a beam of length 'L'

\[ \int_0^L f(x) \delta(x - x_0) \, dx = f(x_0) \quad 0 < x_0 < L \quad (4.2.4a) \]

and

\[ \int_0^L f(x) \delta(x - x_0) \, dx = 0 \quad x_0 < 0, \ x_0 > L \quad (4.2.4b) \]

From equation (4.2.1), (4.2.2) and (4.2.3) we get

\[ EI \frac{\partial^4 y(x,t)}{\partial x^4} + \rho A1 \frac{\partial^2 y(x,t)}{\partial t^2} = \left[ Mg - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 y(\beta,t) \right] \delta(x - \beta) \quad (4.2.5) \]

where \( \beta = vt \) and \( \mu \mu = \rho A1 \) mass per unit length of the beam

\( v = \) velocity of the moving mass.

In Fig.4.2.1

\( x = \) Distance from the fixed end to the point under consideration 'z'.

Assuming the solution of equation (4.2.5) in a series form i.e.

\[ y(x,t) = \sum_{n=1}^{\infty} y_n(x) T_n(t) \quad (4.2.6) \]
Here $y_n(x)$ is the eigen-function of the beam (without M) with the same boundary condition at nth. mode.

To find out $Y_n(x)$ the equation will be

$$Y''_n(x) - \gamma^4_n Y_n(x) = 0 \quad (4.2.7)$$

Where $\gamma^4_n = \frac{\rho A}{E I}$

$\omega_n, n = 1, 2, 3, \ldots \ldots \ldots$ are the natural frequencies of the beam.

The general solution of equation (4.2.7) can be written as

$$Y_n(x) = a \sin(\gamma_n x) + b \cos(\gamma_n x) + c \sinh(\gamma_n x) + d \cosh(\gamma_n x) \quad (4.2.8)$$

Where $a, b, c$ & $d$ are the constant coefficients, depending upon the boundary conditions.

$T_n(t)$ are the function of time are to be calculated

Re-writing the right-hand side of equation (4.2.5) as

$$M g - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 \gamma(\beta, t) \delta(x - \beta) = \sum_{n=1}^{\infty} Y_n(x) S_n(t) \quad (4.2.9)$$

Substituting equation (4.2.6) into equation (4.2.9), equation (4.2.9) transforms to

$$M g - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 \left( \sum_{n=1}^{\infty} Y_n(\beta) T_n(t) \right) \delta(x - \beta) = \sum_{n=1}^{\infty} Y_n(x) S_n(t) \quad (4.2.10)$$

multiplying both side of equation (4.2.10) by $Y_p(x)$ and integrating over the beam length we get

$$\int_0^L M g Y_p(x) \delta(x - \beta) dx - M \sum_{n=1}^{\infty} \int_0^L Y_p(x) \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) \delta(x - \beta) dx$$

$$= \sum_{n=1}^{\infty} \int_0^L Y_p(x) Y_n(x) S_n(t) dx \quad (4.2.11)$$

From equation 4.2.4a and 4.2.4b
\[ Mg \int_0^L Y_p(x) \delta(x - \beta) \, dx = Mg Y_p(\beta) \]  
\[ \text{(4.2.12)} \]

and

\[ Mg \sum_{n=1}^{\infty} \int_0^L Y_p(x) \left( \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) \right) \delta(x - \beta) \, dx \]

\[ = M \sum_{n=1}^{\infty} \left( \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) \right) \int_0^L Y_p(x) \, \delta(x - \beta) \, dx \]

\[ = M \sum_{n=1}^{\infty} \left( \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) \right) Y_p(\beta) \]  
\[ \text{(4.2.13)} \]

From orthogonality principle and from the orthogonal properties of the function \( Y_n(x) \), right side of equation (4.2.11) i.e.

\[ \sum_{n=1}^{\infty} \int_0^L Y_p(x) Y_n(x) S_n(t) \, dx = \sum_{n=1}^{\infty} S_n(t) \int_0^L Y_p(x) Y_n(x) \, dx \]

\[ = S_p(t) \int_0^L Y_p(x) Y_p(x) \, dx + S_1(t) \int_0^L Y_p(x) Y_1(x) \, dx + \ldots \]

\[ = S_p(t) \int_0^L Y_p(x) Y_p(x) \, dx \quad \{ \text{All other terms vanish} \} \]

\[ = S_p(t) V_p = V_p S_p(t) \]  
\[ \text{(4.2.14)} \]

[ As \( \int_0^L Y_p(x) Y_n(x) \, dx = \begin{cases} 0, & n \neq p \\ V_p, & n = p \end{cases} \)]

Using (4.2.12), (4.2.13), (4.2.14) the equation (4.2.11) reduces to

\[ Mg Y_p(\beta) - M \sum_{n=1}^{\infty} \left( \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) \right) Y_p(\beta) = S_p(t) V_p \]  
\[ \text{(4.2.15)} \]

Equation (4.2.15) is re-written as

\[ S_p(t) = \frac{M}{V_p} \left[ g - \sum_{n=1}^{\infty} \left( \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 Y_n(\beta) T_n(t) \right) \right] Y_p(\beta) \]  
\[ \text{(4.2.16)} \]

From equation (4.2.5) and (4.2.9) we have
Combining equation (4.2.16) and (4.2.17) we get,

\[
\mathcal{E} \frac{\partial^4 y(x,t)}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y(x,t)}{\partial t^2} = \sum_{n=1}^{\infty} Y_n(x) S_n(t) \tag{4.2.17}
\]

From equation (4.2.6) and (4.2.18) we have

\[
\mathcal{E} \frac{\partial^4}{\partial x^4} \left( \sum_{n=1}^{\infty} Y_n(x) T_n(t) \right) + \rho_1 A_1 \frac{\partial^2}{\partial x^2} \left( \sum_{n=1}^{\infty} Y_n(x) T_n(t) \right) = \sum_{n=1}^{\infty} Y_n(x) M \left[ g - \sum_{q=1}^{\infty} \left( \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial \beta} \right)^2 Y_q(\beta) T_q(t) \right] Y_n(\beta) \tag{4.2.18}
\]

Equation (4.2.19) is re-written as

\[
\mathcal{E} \sum_{n=1}^{\infty} Y_n^{iv}(x) T_n(t) + \rho_1 A_1 \sum_{n=1}^{\infty} Y_n(x) T_{n,n}(t) = \sum_{n=1}^{\infty} Y_n(x) M \left[ g - \sum_{q=1}^{\infty} \left( \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial \beta} \right)^2 Y_q(\beta) T_q(t) \right] Y_n(\beta) \tag{4.2.19}
\]

By substituting the value of \( Y_n^{iv}(x) \) from (4.2.7) into equation (4.2.20) we get;

\[
\mathcal{E} \sum_{n=1}^{\infty} Y_n^{iv}(x) T_n(t) + \rho_1 A_1 \sum_{n=1}^{\infty} Y_n(x) T_{n,n}(t) = \sum_{n=1}^{\infty} Y_n(x) M \left[ g - \sum_{q=1}^{\infty} \left( \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial \beta} \right)^2 Y_q(\beta) T_q(t) \right] Y_n(\beta) \tag{4.2.21}
\]

Equation (4.2.21) can be re-written as
Fig. 4.2.2. Displacement at the point \( z(\text{m.}) \) vs. time (Sec.). Mild Steel beam \((1.2 \times 0.05 \times 0.006 \text{ m.})\), \( M = 1 \text{ kg.}, v = 20 \text{ KMPH} \).

Fig. 4.2.3. Displacement at the point \( z(\text{m.}) \) vs. time (Sec.). Mild Steel beam \((1.2 \times 0.05 \times 0.006 \text{ m.})\), \( M = 1 \text{ kg.}, v = 80 \text{ KMPH} \).
Fig. 4.2.4. Displacement at the point $z$(m.) vs. time(Sec.). Mild Steel beam ($1.2 \times 0.05 \times 0.006 \text{ m.}$), $M = 1 \text{ kg.}$, $v = 150 \text{ KMPH.}$

Fig. 4.2.5. Displacement at the point $z$(m.) vs. time(Sec.). Mild Steel beam ($1.2 \times 0.05 \times 0.006 \text{ m.}$), $M = 2 \text{ kg.}$, $v = 20 \text{ KMPH.}$
Fig. 4.2.6. Displacement at the point \( z(\text{m.}) \) vs. \( \text{time (Sec.)} \). Mild Steel beam \((1.2 \times 0.05 \times 0.006 \text{ m.})\), \( M = 2 \text{ kg.} \), \( v = 80 \text{ KMPH} \).

Fig. 4.2.7. Displacement at the point \( z(\text{m.}) \) vs. \( \text{time (Sec.)} \). Mild Steel beam \((1.2 \times 0.05 \times 0.006 \text{ m.})\), \( M = 2 \text{ kg.} \), \( v = 150 \text{ KMPH} \).
M = 1200 gms., Mild Steel beam (1.5 x 0.05 x 0.006 m.).

\( v = 40.0 \text{ KMPH} \), Mild Steel beam (1.5 x 0.05 x 0.006 m.).

Displacement at the end point (cm.)

A

86
Fig. 4.2.10. 3-D graph for displacement at the end point (cm.)-mass of the moving body (gms.) - time (Sec.),
v = 180.0 KMPH., Mild Steel beam (1.5 x 0.05 x 0.006 m.).
\[
\sum_{n=1}^{\infty} Y_n(x) \left[ E \frac{\partial^4 T_n(t)}{\partial x^4} + \frac{\partial v}{\partial t} \frac{\partial^2 Y_n(\beta)}{\partial \beta^2} T_q(t) \right] Y_n(\beta) = 0
\] (4.2.22)

As the equation (4.2.22) must satisfy, for any arbitrary value of \(x\), so

\[
E \frac{\partial^4 T_n(t)}{\partial x^4} + \frac{\partial v}{\partial t} \frac{\partial^2 Y_n(\beta)}{\partial \beta^2} T_q(t) = 0
\] (4.2.23)

Equation (4.2.23) is a differential equation of 2nd order, which is solved by Runge-Kutta method.

**4.2.2. NUMERICAL ANALYSIS**

For the numerical results of dynamic response, the following beam specimen and following specific locations on the beam are selected.

I) Beam Specimens: (Mild Steel Cantilever)
   a) Length = 1.2m., Breadth = 0.05m., Width = 0.006m.
   b) Length = 1.5m., Breadth = 0.05m., Width = 0.006m.

II) Moving Mass Considered
   a) \(M = 1.0\) kg.
   b) \(M = 2.0\) kg.

III) Velocity of the Moving Mass Considered
   a) \(v = 20\) KMPH.
b) \( v = 80 \text{ KMPH} \)
c) \( v = 150 \text{ KMPH} \)
d) \( v = 180 \text{ KMPH} \)

IV) Location Selected
a) \( x = \text{‘End Point’} \)
b) \( x = vt \)

The numerical results are found out using the above expressions i.e. eq.no. 4.2.6 and 4.2.23 and the corresponding computer programming as given in Appendix- IIA. All these numerical results are presented in Fig. 4.2.2 to 4.2.10.

For solving the equation (4.2.6), the values of \( T_n(t) \) are obtained at required position from equation (4.2.23), using Runge-Kutta technique. Numerical calculation are carried out in a computer using first six modes of vibration amplitudes in equation (4.2.6) for convergence of the result with accuracy of order \( 10^{-4} \) units.

In Figs. 4.4.2 to 4.2.4, dynamic deflection at the free end point of the cantilever beam and at \( x = vt \) of the beam have been plotted with time, for various velocities of the moving mass, \( M = 1.0 \text{ kg} \). In Figs 4.2.5 to 4.2.7, similar results have been plotted for the other moving mass, \( M = 2 \text{ kg} \). In Fig.4.2.8 a three dimensional graph has been plotted to present the dynamic deflection at the free end of the beam with the velocity of the moving mass vs. time. In Fig.4.2.9 and Fig.4.2.10, similar graphs have also been plotted for dynamic deflection at the free end, for variation of mass of the moving body with time.
4.2.3. DISCUSSION

(1) From Figures 4.2.2 to 4.2.7, it is observed that as the velocity of the moving mass increases, the deflection at the end point of the beam decreases. This is because at higher velocity of the moving mass, lower modes are not excited, which mainly contribute for larger dynamic displacement of the beam. As the lower modes of vibration contribute larger amplitude as compared to that of higher modes, dynamic displacement of the beam decreases.

(2) It is noticed from Figures 4.2.3, 4.2.4, 4.2.6 and 4.2.7 that at the free end dynamic deflection of the beam decreases sharply with time and again increases at higher mass velocities because of higher modes of vibration are more prominent than that of 1st mode. However at 'x = vt', as the moving mass is on the point where the dynamic deflection is to be recorded, the dynamic deflection mainly depends upon the weight of the moving mass, whereas at the end point of the beam i.e. at 'x=L', the dynamic deflection mainly depends on the modal vibration of the beam. Therefore dynamic deflection exhibits sharp changes in magnitude as well as direction at the free end.

(3) Similar conclusions can be drawn from 3-D graphs presented in Fig 4.2.8 to 4.2.10. Fig.4.2.8 shows the variation of the displacements at the end point of the beam for different velocities of the moving mass in the region 31 KMPH. to 120 KMPH. It is observed from the figure, that as the velocity of the moving mass increases, the deflection at the end point of the beam decreases. Figs.4.2.9 and 4.2.10 show the variation of displacements at the end point of the beam for different magnitude of the mass of the moving body. From those figures, it is found that as the mass of the moving body increases, the end deflection also increases. This is due to increase in the inertia of the moving mass.
Dynamic response of a cantilever beam with a transverse crack as well as with a moving mass has been developed using the theoretical analysis as detailed below along with numerical results.

4.3.1. THEORETICAL ANALYSIS

The equation of motion of an uniform beam of mass 'm' per unit length, subjected to a moving mass 'M', as shown in Fig. 4.3.1 can be written as [100,104,125,126];

\[ E I \frac{\partial^4 y(x,t)}{\partial x^4} + \rho 1 A 1 \frac{\partial^2 y(x,t)}{\partial t^2} = \left[ M g - M \frac{\partial^2 y(\beta,t)}{\partial t} \right] \delta(x - \beta) \]  (4.3.1)

Neglecting damping in the System,

where

t = time taken by the moving mass to travel a distance \( \beta \) from the fixed end of the cantilever beam

x = Distance from the fixed end to the point under consideration 'z', the point of interest where the deflection of the beam is considered ('z' it may be anywhere on the beam).

Assuming the solution of equation(4.3.1) in a series form i.e.

\[ y(x,t) = \sum_{n=1}^{\infty} y_n(x) T_n(t) \]  (4.3.2)

Here \( y_n(x) \) is the eigen-function of the beam (without M) with the same boundary condition at nth. mode.

To find out \( Y_n(x) \) the equation will be
Fig. 4.3.1 Beam with moving mass

z is the point of interest where the displacement of the beam is to be considered.
\[ Y_n''(x) - \gamma_n^4 Y_n(x) = 0 \]  \hspace{1cm} (4.3.3)

Where \( \gamma_n^4 = \frac{\rho A \omega_n^2}{EI} \)

\( \omega_n, n = 1, 2, 3 \ldots \) are the natural frequencies of the beam.

The general solution of equation (4.3.3) can be written as

\[ Y_n(x) = a_n \sin(\gamma_n x) + b_n \cos(\gamma_n x) + c_n \sinh(\gamma_n x) + d_n \cosh(\gamma_n x) \]  \hspace{1cm} (4.3.4a)

For \( 0 \leq x \leq L_1 \)

\[ Y_n(x) = a_2 \sin(\gamma_n x) + b_2 \cos(\gamma_n x) + c_2 \sinh(\gamma_n x) + d_2 \cosh(\gamma_n x) \]  \hspace{1cm} (4.3.4b)

For \( L_1 < x \leq L \)

Where \( a_1, b_1, c_1, d_1, a_2, b_2, c_2 \) & \( d_2 \) are the constant coefficients, depending upon the boundary conditions, and can be derived from Chapter 3 (Section 3.2).

And \( T_n(t) \) are the function of time are to be calculated.

Re-writing the right-hand side of equation (4.3.3) as

\[
\left[ M g - M \frac{\partial^2 \gamma(\beta, t)}{\partial t^2} \right] \delta(x - \beta) = \sum_{n=1}^{\infty} Y_n(x) S_n(t) \]  \hspace{1cm} (4.3.5)

Proceeding in the same manner as Section 4.2, Parhi and Behera [104], the final equation can be obtained as;

\[
E I Y_n''(t) + \rho A T_{n,t}(t) = \frac{M}{V_n} \left[ g - \sum_{q=1}^{\infty} Y_q(\beta) T_{a,nn}(t) \right] Y_n(\beta) = 0 \]  \hspace{1cm} (4.3.6)
Fig. 4.3.2. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.2 kg., v = 15 KMPH.

Fig. 4.3.3. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.2 kg., v = 31 KMPH.
Fig. 4.3.4. Displacement at the end point (m.) vs. time (Sec.) Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.2 kg., v = 60 KMPH.

Fig. 4.3.5. Displacement at the end point (m.) vs. time (Sec.) Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.2 kg., v = 180 KMPH.
Fig. 4.3.6. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.5 x 0.05 x 0.006 m.), $M = 1.848$ kg., $v = 15$ KMPH.

Fig. 4.3.7. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.5 x 0.05 x 0.006 m.), $M = 1.848$ kg., $v = 31$ KMPH.
Fig. 4.3.8. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.848 kg., v = 60 KMPH.

Fig. 4.3.9. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.848 kg., v = 180 KMPH.
Equation (4.3.6) is a differential equation of 2nd order, which is solved by Runge-Kutta method.

4.3.2 NUMERICAL ANALYSIS

For the numerical results of dynamic response the following beam specimens and the specific locations on the beam where the response is determined are selected.

I) Beam Specimen: (Mild Steel Beam)
   Length = 1.5m., Breadth = 0.05m., Width = 0.006m.

II) Moving Mass Considered:
   (a) \( M = 1.20 \, \text{kg.} \)
   (b) \( M = 1.848 \, \text{kg.} \)

III) Velocity of the Moving Mass Considered:
   (a) \( v = 15 \, \text{KMPH} \)
   (b) \( v = 31 \, \text{KMPH} \)
   (c) \( v = 60 \, \text{KMPH} \)
   (d) \( v = 180 \, \text{KMPH} \)

IV) Location Selected:
   \( x = \text{‘End Point’} \)

V) Transverse Crack
   Depth = 0.003m., at a distance 0.1m. from fixed end., crack width = B (Same as beam width)
Equation (4.3.1) is solved using (4.3.2). The values of $T_n(t)$, are obtained at required position from equation (4.3.6), using Runge-Kutta method. First six modes of vibration are taken into account, for convergence of results to an order of $10^{-4}$ units.

Fig.4.3.2 to Fig.4.3.9 show the variation of displacement at the free end of cracked as well as uncracked beams with variation of time for different velocities of the moving masses 1.2 kg. and 1.848kg. using computer programming given in Appendix-IIb.

### 4.3.3 DISCUSSION

From the numerical analysis, the following points have been discussed;

(1) It is evident from Figs. 4.3.2 to 4.3.9, that as the velocity of the moving mass increases, the deflection at the end point of the beam decreases. This is because at higher speed, lower modes are not in resonance, which mainly contribute for larger transverse amplitude of the beam.

(2) It is also, found that as the mass of the moving body increases, the end deflection of the cracked beam increases relative to that of the corresponding uncracked beam. But in both cases, the deflection increases as the distance of moving mass from the fixed end increases.
4.4 DYNAMIC RESPONSE OF CRACKED SHAFT SUBJECTED TO MOVING MASS

Dynamic response of a simply supported shaft with a transverse crack as well as with a moving mass has been developed using the theoretical analysis as detailed below along with its numerical results.

4.4.1. THEORETICAL ANALYSIS

The equation of motion of an uniform shaft of mass \( m \) per unit length subjected to a moving mass \( M \) as shown in Fig. 4.4.1, can be written [100,125,126] as:

\[
\begin{align*}
&EI \frac{d^4y(x,t)}{dx^4} + \mu \frac{d^2y(x,t)}{dt^2} \\
&-\rho l \left[ 1 + \frac{E}{kkG} \frac{d^4y(x,t)}{dx^2dt^2} + \frac{\rho l^2}{kkG} \frac{d^4y(x,t)}{dt^4} \right] = P(x,t)
\end{align*}
\] (4.4.1)

Neglecting the system damping

The External force \( P(x,t) \) can be taken as

\[
P(x,t) = F(t) \delta(x-vt)
\] (4.4.2)

\[
F(t) = Mg - M \left( \frac{\partial}{\partial \beta} + v \frac{\partial}{\partial \beta} \right)^2 y(\beta,t)
\] (where, \( \beta = vt \) (4.4.3)

Where \( F(t) \) is the force due to moving mass at that instant \( \delta \) is the Dirac-delta function.
Fig. 4.4.1a Cracked shaft with moving mass
z is the point of interest where the displacement of the shaft is to be considered.

Fig. 4.4.1b Cross Sectional View of the cracked shaft.

Fig. 4.4.1c Simply supported cracked shaft with various coupling forces.
From equation (4.4.1), (4.4.2) and (4.4.3) we get
\[ E\frac{\partial^4 y(x,t)}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y(x,t)}{\partial t^2} - \rho_1 l \left[ 1 + \frac{E}{k k G} \frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} + \frac{\rho_1 l}{k k G} \frac{\partial^4 y(x,t)}{\partial t^4} \right] \]
\[ = \left[ Mg - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 y(\beta,t) \right] \delta(x - \beta) \]  
(4.4.4)

where \( \beta = vt \) and \( \mu_\mu = \rho_1 A_1 \) mass per unit length of the beam

\( v = \) velocity of the moving mass.

\( x = \) Distance from the fixed end to the point under consideration 'z'.

Assuming the solution of equation (4.4.4) in a series form i.e.

\[ y(x,t) = \sum_{n=1}^{\infty} y_n(x) T_n(t) \]  
(4.4.5)

Where \( T_n(t) \) \( n = 1,2,3 \ldots \ldots \ldots \ldots \) are functions of time;

Here \( y_n(x) \) are the eigen-functions of the shaft (without M) and are obtained from

\[ E\frac{\partial^4 y(x,t)}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y(x,t)}{\partial t^2} - \rho_1 l \left[ 1 + \frac{E}{k k G} \frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} + \frac{\rho_1 l}{k k G} \frac{\partial^4 y(x,t)}{\partial t^4} \right] = 0 \]  
(4.4.6)

The expressions for \( Y_n(x) \) are derived from Chapter-3 (Section 3.3), i.e.

\[ Y_n(x) = (\bar{Y}_1(\bar{x})) L \begin{cases} 0 \leq \bar{x} \leq L_1 \\ \text{before crack region} \end{cases} \]  
(4.4.7a)

\[ Y_n(x) = (\bar{Y}_2(\bar{x})) L \begin{cases} L_1 < \bar{x} \leq L \\ \text{after crack region} \end{cases} \]  
(4.4.7b)

The right side of equation (4.4.4) can be written as

\[ \left[ Mg - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 y(\beta,t) \right] \delta(x - \beta) = \sum_{n=1}^{\infty} y_n(x) S_n(t) \]  
(4.4.8)
Fig. 4.4.2. Displacement (cm.) at the point $z$ vs. time (sec.) for $v = 70$ KMPH., $M = 1.0$ kg., mild steel shaft, $L = 1.5$ m., $L_1 = 0.2$ m., $R = 0.01$ m., $a_1/D = 0.4$.

Fig. 4.4.3. Displacement (cm.) at the point $z$ vs. time (sec.) for $v = 110$ KMPH., $M = 1.0$ kg., mild steel shaft, $L = 1.5$ m., $L_1 = 0.2$ m., $R = 0.01$ m., $a_1/D = 0.4$. 
Fig. 4.4.4. Displacement (cm.) at the point z vs. time (sec.) for 
v = 160 KMPH., M = 1.0 kg., mild steel shaft, L = 1.5 m., 
L₁ = 0.2 m., R = 0.01 m., a₁/D = 0.4.

Fig. 4.4.5. Displacement (cm.) at the point z vs. time (sec.) for 
v = 70 KMPH., M = 2.0 kg., mild steel shaft, L = 1.5 m., 
L₁ = 0.2 m., R = 0.01 m., a₁/D = 0.4.
Fig. 4.4.6. Displacement (cm.) at the point z vs. time (sec.) for $v = 110$ KMPH., $M = 2.0$ kg., mild steel shaft, $L = 1.5$ m., $L_1 = 0.2$ m., $R = 0.01$ m., $a_1/D = 0.4$.

Fig. 4.4.7. Displacement (cm.) at the point z vs. time (sec.) for $v = 160$ KMPH., $M = 2.0$ kg., mild steel shaft, $L = 1.5$ m., $L_1 = 0.2$ m., $R = 0.01$ m., $a_1/D = 0.4$. 
Fig. 4.4.8. Displacement (cm.) at the point $z$ vs. time (sec.) for $v = 80$ KMPH., $M = 1.5$ kg., mild steel shaft, $L = 1.5$ m., $L_1 = 0.2$ m., $R = 0.01$ m., $a_1/D = 0.4$.

Fig. 4.4.9. Displacement (cm.) at the point $z$ vs. time (sec.) for $v = 120$ KMPH., $M = 1.5$ kg., mild steel shaft, $L = 1.5$ m., $L_1 = 0.2$ m., $R = 0.01$ m., $a_1/D = 0.4$. 

106
Displacement (cm.) at the point z vs. time (sec.) for $v = 180$ KMPH., $M = 1.5$ kg., mild steel shaft, $L = 1.5$ m., $L_t = 0.2$ m., $R = 0.01$ m., $a1/D = 0.4$. 

Fig. 4.4.10.
Fig. 4.4.11a. DEFLECTION OF THE SHAFT UNDER THE MOVING MASS FOR VARIOUS MASS RATIO (MR = M / Mass of the Shaft) 
\[v = 61 \text{ KMPH}, \, L = 2.2 \text{ m}, \, R = 0.01\text{m.}\] 
Mild steel shaft.

Fig. 4.4.11b. DEFLECTION OF THE BEAM UNDER THE MOVING MASS FOR VARIOUS MASS RATIO (DONE BY LEECH et al. [77]).
Now substituting (4.4.5) into (4.4.8), equation (4.4.8) transforms to;

\[
Mg - M \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 \sum_{n=1}^{\infty} y_n(\beta) T_n(t) \right] \delta(x - \beta) = \sum_{n=1}^{\infty} y_n(x) S_n(t) \quad (4.4.9)
\]

Proceeding in the same manner just as section 4.2, the expression for finding the value of \(T_n(t)\) are obtained as;

\[
\begin{align*}
\frac{\partial^4}{\partial x^4} \left( \sum_{n=1}^{\infty} Y_n(x) T_n(t) \right) + \rho l A_1 \frac{\partial^2}{\partial t^2} \left( \sum_{n=1}^{\infty} Y_n(x) T_n(t) \right) \\
- \rho \Gamma \left[ 1 + \frac{E}{kk G} \frac{\partial^4}{\partial x^4} \sum_{n=1}^{\infty} Y_n(x) T_n(t) \right] + \frac{\rho l^2 \Gamma}{kk G} \frac{\partial^6}{\partial t^6} \left( \sum_{n=1}^{\infty} Y_n(x) T_n(t) \right) \\
= \sum_{n=1}^{\infty} Y_n(x) \frac{M}{V_n} \left[ g - \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial \beta} \right)^2 y_n(\beta) T_n(t) \right] y_n(\beta) \quad (4.4.10)
\end{align*}
\]

Equation (4.4.10) is a fourth order differential equation, which is solved by the Runge-Kutta method and the values of \(T_n(t)\) are found out. Putting the values of \(T_n(t)\) and \(Y_n(x)\) in equation (4.4.5) the amplitude of dynamic deflection \(y(x,t)\) at any desired point is calculated.

4.4.2 NUMERICAL ANALYSIS

For numerical analysis a mild steel shaft of uniform cross section having a transverse crack, subjected to moving mass as referred in theory is considered. The dimensions of the shaft and the moving mass for our analysis are taken as;
I) Shaft: (Mild Steel Simply supported shaft)
Length (L) = 1.5m.
Radius (R) = 0.01m.

II) Crack:
Relative Crack Depth (a1/D) = 0.4
Crack Width (2b) = 0.0196 m.
Crack distance from left end (L1) = 0.2m.

III) Moving Mass
a) M = 1.0kg.
b) M = 1.5kg.
c) M = 2.0kg.

The numerical results for dynamic displacement at point 'z' are obtained using equation(4.4.5) along with the computer programming given in Appendix-IIIc. Graphs are plotted to represent the variations of displacement at the mid point of the shaft (i.e. x = L/2) as well as on the moving mass (i.e.x=vt) with increase in time for different velocities of the moving mass. They are shown in Fig. 4.4.2. to Fig.4.4.7 . From Fig.4.4.8 to Fig.4.4.10 graphs are plotted for displacements at the mid point of the shaft (i.e. x =L/2) for both cracked and uncracked shafts, with a moving mass but for different velocities of the mass in order to compare the results.

For solving the equation(4.4.5), the values of Tn(t) are obtained at required positions from equation(4.4.10), using Runge-Kutta technique. Numerical calculations are carried out on a computer, using first six modes of vibration amplitudes in equation(4.4.5) for convergence of the result to an order of 10^-4 units.
4.4.3. DISCUSSION

(1) From Fig.4.4.8 to Fig.4.4.10, appreciable variations of dynamic response are observed between the cracked and uncracked shaft.

(2) It is also noticed from Figs.4.4.2. to 4.2.7, that the dynamic deflection of the cracked shaft, at the position, $x = vt$ for higher velocity of moving mass is somewhat low, in comparison to that for lower velocities.

(3) From Fig. 4.4.11 (a) & (b) it is found that the results obtained from the theory developed for uncracked shaft and the results as presented earlier by Leech et al.[77] are tallying quite well which authenticate the theory developed.

4.5. EXPERIMENTAL ANALYSIS

In order to establish the theory developed, experimental analysis is essential for the current investigation. For the experimental analysis an experimental set up has been developed in order to measure dynamic deflection of cantilever cracked/uncracked beam with moving mass but not for the shafts. The details of the set-up and experimental procedure along with results are highlighted below.

4.5.1. EXPERIMENTAL SET-UP

The experimental set-up is used for performing the experiments, is shown in the schematic diagram (Fig.4.5.1). For conducting the experiments the equipment required are listed below.
Photo Graph 2. Experimental set-up of cracked cantilever beam with moving mass.
Fig. 4.5.1. Schematic block diagram of experimental set-up.
Fig. 4.5.2. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.2 x 0.05 x 0.006 m.), M = 1.2 kg., v = 20 KMPH.

Fig. 4.5.3. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.2 x 0.05 x 0.006 m.), M = 1.2 kg., v = 40 KMPH.
Fig. 4.5.4. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.2 x 0.05 x 0.006 m.), $M = 1.2$ kg., $v = 60$ KMPH.

Fig. 4.5.5. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.2 x 0.05 x 0.006 m.), $M = 2.0$ kg., $v = 20$ KMPH.
Fig. 4.5.6. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.2 x 0.05 x 0.006 m.), M = 2.0 kg., v = 40 KMPH.

Fig. 4.5.7. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.2 x 0.05 x 0.006 m.), M = 2.0 kg., v = 60 KMPH.
Fig. 4.5.8. Displacement at the end point (m.) vs. time (Sec.) Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.2 kg., v = 15 KMPH.

Fig. 4.5.9. Displacement at the end point (m.) vs. time (Sec.) Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.2 kg., v = 40 KMPH.
Fig. 4.5.10. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.848 kg., v = 40 KMPH.

Fig. 4.5.11. Displacement at the end point (m.) vs. time (Sec.). Mild Steel beam (1.5 x 0.05 x 0.006 m.), M = 1.848 kg., v = 60 KMPH.
<table>
<thead>
<tr>
<th>No.</th>
<th>Items</th>
<th>Specifications</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LVDT (Linearly Variable Displacement Transducer)</td>
<td>Range - (+/-) 50 mm.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Model - SI 7-732/2550</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Operating temp. +10°C to 50°C</td>
</tr>
<tr>
<td>2</td>
<td>Motor</td>
<td>Type-M-32</td>
</tr>
<tr>
<td></td>
<td></td>
<td>230V, 50Hz., 0.5 Amp, 0.5 h.p.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>R.P.M.-1400, Make-SUR Electrical(P)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ltd., Cal-15, Sl.No.-7353559</td>
</tr>
<tr>
<td>3</td>
<td>Variac</td>
<td>Type 569/2510</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Input - 230V, 50-60Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Output- 0-270V</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximum Load 8 Amp.</td>
</tr>
<tr>
<td>4</td>
<td>Digital Displacement Indicator</td>
<td>Model SI-74, Sl.No. 1227</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 Volt RMS, 3.8 K. Hz.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5 Channels, 5.5 Kg.</td>
</tr>
<tr>
<td>5</td>
<td>Extension board</td>
<td>230-240 AC supply</td>
</tr>
<tr>
<td>6</td>
<td>AC Supply</td>
<td>1.2, 1.848, 2.0 Kg.</td>
</tr>
<tr>
<td>7</td>
<td>Clamping Device</td>
<td>(a) Uncracked Type (1.2 x 0.05 x 0.006 m.)</td>
</tr>
<tr>
<td>8</td>
<td>Moving Mass</td>
<td>(b) Cracked (1.5 x 0.05 x 0.006 m.)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Crack position from fixed end (L1) = 0.1m.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Crack depth (a1) = 0.003m.</td>
</tr>
<tr>
<td>9</td>
<td>Beam</td>
<td>(a) Uncracked Type (1.2 x 0.05 x 0.006 m.)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) Cracked (1.5 x 0.05 x 0.006 m.)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Crack position from fixed end (L1) = 0.1m.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Crack depth (a1) = 0.003m.</td>
</tr>
</tbody>
</table>
4.5.2. EXPERIMENTAL PROCEDURE

A number of tests are conducted on mild steel beam specimen (1.2 x 0.05 x 0.006 m., uncracked) and (1.2 x 0.05 x 0.006 m., with transverse crack 0.003m. depth, at a distance L₁=0.1m. from the fixed end). The moving masses taken for experiment are 1.2, 1.848 and 2.0 Kg. and velocity induced during experiments are 15, 20, 40 and 60 KMPH.

Experimental results for dynamic response of the free end of cantilever beam specimens for various positions of the moving mass are recorded by positioning a LVDT as shown in the Fig. 4.5.1. The LVDT is so arranged in the measuring system that it only records the instantaneous displacement of the beam in the digital displacement indicator, as the moving mass reaches a predetermined location on the cantilever beam specimen.

4.5.3. EXPERIMENTAL RESULTS

Experimental results for dynamic deflection of cantilever beams with crack and without crack are obtained using the above experimental set-up and procedure.

For uncracked cantilever beam results are plotted in Fig. 4.5.2 to 4.5.7 for different velocities of the moving mass (i.e. v = 20, 40 and 60 KMPH.) and for different magnitudes of the moving mass (i.e. M=1.2 kg.). Corresponding numerical results are also presented in the same graph for comparison.
For cracked cantilever beam, similar results are plotted in Figs. 4.5.8 to 4.5.11 for different velocities of the moving mass (i.e. \( v = 15, 40 \) and 60 KMPH) and for different magnitudes of the moving mass (i.e. \( M = 1.2, 1.848 \) kg.). Corresponding numerical results are also presented in the same graph for comparison.

While conducting the experiments, sufficient precautions have been taken in order to achieve the best possible accuracy in measurements.

**4.6 DISCUSSION AND CONCLUSIONS**

From the above theoretical investigations and comparisons of numerical and experimental results, the following salient points have been discussed and major conclusions have been drawn. They are highlighted below.

**4.6.1 DISCUSSION**

1. From Figures 4.2.2 to 4.2.8, it is observed that as the velocity of the moving mass increases, the deflection at the end point of the beam decreases. This is because at higher velocity of the moving mass, lower modes are not excited, which mainly contribute for larger dynamic displacement of the beam. As the lower modes of vibration contribute larger amplitude as compared to that of higher modes, dynamic displacement of the beam decreases.

The above phenomenon can also be visualised from Figs. 4.3.2 to 4.3.9, and Fig. 4.4.2. to Fig. 4.4.7, that as the velocity of the moving mass increases, the deflection at the end point of the beam decreases.
(2) It is noticed from Figures 4.2.3, 4.2.4, 4.2.6 and 4.2.7 that at the free end dynamic deflection of the beam decreases sharply with time and again increases at higher mass velocities because of higher modes of vibration are more prominent than that of 1st mode. However at 'x = vt', as the moving mass is on the point where the dynamic deflection is to be recorded, the dynamic deflection mainly depends upon the weight of the moving mass, whereas at the end point of the beam i.e. at 'x=L', the dynamic deflection mainly depends on the modal vibration of the beam. Therefore dynamic deflection exhibits sharp variation in magnitude as well as direction at the free end.

(3) From Figs.4.2.9 and 4.2.10, it is found that as the mass of the moving body increases, the end deflection increases. This is due to increase in the inertia of the moving mass.

(4) From Figs.4.3.2 to 4.3.9, it is also found that as the mass of the moving body increases, the end deflection of the cracked beam increases relative to that of the corresponding uncracked beam. But in both cases, the deflection increases as the distance of the moving mass from the fixed end increases.

(5) Appreciable variation in of dynamic deflection of the shaft with a moving mass is observed between the cracked and uncracked cases as per Fig.4.4.8 to Fig.4.4.10. It is also noticed from Fig.4.4.2 to Fig.4.4.7, that the dynamic deflection of the cracked shaft, on the moving mass (i.e. x = vt) for higher velocity is lower, in comparison to that at lower velocities.

(6) In Figs. 4.4.11 (a) and (b) comparison has been made for uncracked shaft between the results obtained from the theory developed and the results presented by Leech et al.[77]. The above comparison shows good agreement between them.
From the experimental results presented in Figs. 4.5.2 to 4.5.11 and the corresponding numerical results plotted in the same figures, it has been observed that they tally well and show similar trend as discussed above.

4.6.2. CONCLUSIONS

From the theoretical and experimental analysis as well as from the above discussions enumerated above, the following conclusions are drawn.

1. The deflection of the beam structure mainly depends upon the velocity of the moving mass. As the moving mass velocity increases, the dynamic deflection of the beam decreases, which is a peculiar phenomenon. However, as the magnitude of the moving mass increases, the deflection increases.

2. It is also evident from theoretical and experimental analysis that presence of crack in structures makes significant difference in its dynamic deflection, to that of uncracked one when subjected to moving load.

3. The present study may be extended to provide information on the dynamic design of complicated structures used in actual practice subjected to a moving load. This will also help in preventing sudden failure of such shafts and other similar structures.